RECOGNITION OF SEIFERT FIBERED SPACES WITH BOUNDARY IS IN NP

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Abstract. We show that the decision problem of recognising whether a triangulated 3-manifold admits a Seifert fibered structure with non-empty boundary is in NP. We also show that the problem of producing Seifert data for a triangulation of such a manifold is in the complexity class FNP. We do this by proving that in any triangulation of a Seifert fibered space with boundary there is both a fundamental horizontal surface of small degree and a complete collection of normal vertical annuli whose total weight is bounded by an exponential in the square of the triangulation size.

Contents

1. Introduction

A basic question in low-dimensional topology is 3-manifold homeomorphism: given two 3-manifolds, decide whether they are homeomorphic. This recognition problem is decidable as a consequence of Perelman's proof of Thurston's geometrisation conjecture (several proofs have been given of this; see [29] for an overview). This is in contrast to the n-manifold case for $n \geq 4$, where the problem has been shown to be undecidable [19].

The complexity of this problem, however, is not very well understood. Kuperberg showed that there is an algorithm for 3-manifold homeomorphism which has running time bounded by a bounded tower of exponentials – that is, that it is at most of the order of

$$2^{2^{\cdot^{\cdot^{\cdot^{t}}}}}$$

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for some fixed height [15] – where the height is not known except among hyperbolic manifolds [30]. It is also known the problem is at least as hard as finite graph isomorphism [16]. The natural question that follow is whether the problem lies in **NP**. To prove this, we might first hope to prove that recognising hyperbolic and Seifert fibered manifolds is in **NP**.

This paper concerns itself with the Seifert fibered case when the boundary is non-empty. We consider two algorithmic problems. First, the SEIFERT FIBERED SPACE WITH BOUNDARY RECOGNITION decision problem: given a triangulation of some 3-manifold, decide whether it admits a Seifert fibered structure and has non-empty boundary.

Theorem 1.1. *The problem* SEIFERT FIBERED SPACE WITH BOUNDARY RECOGNITION *is in* ***NP****.*

Second, the NAMING SEIFERT FIBERED WITH BOUNDARY problem: given a triangulation of a Seifert fibered 3-manifold with non-empty boundary, output a valid set of Seifert data for it.

Theorem 1.2. *The problem* NAMING SEIFERT FIBERED WITH BOUNDARY *is in* ***FNP****.*

Remark 1.3. The complement of the torus knot $T(p, q)$ is the Seifert fibered space $[D^2, -s/q, -r/p]$ where r/s is a fraction such that $ps - qr = 1$. Baldwin and Sivek showed previously that, given a knot complement, deciding if it is the complement of a torus knot is in **NP** [2]. We can strengthen this to the 3-manifold setting by certifying that the given 3-manifold is an appropriate Seifert fibered space and giving p/q and r/s in addition to the Seifert data.

The main piece of technical machinery we develop is that of *split handle structures*, which are defined in Section 4 and are essential for the results in Section 5. They arise from cutting handle structures along normal surfaces, then requiring that normal surfaces in the resulting manifold are disjoint from the cut-open boundary (the *forbidden region*). We extend the usual normal surface theory to this setting in Appendix A, culminating in Proposition A.16. We use this in Section 4.2 to give some bounds on collections of "relatively" fundamental surfaces. This work is based on ideas used by King [14, §3.2-3] and Lackenby [17, §12.2], among others, which have not been previously rigorously generalised. Appendix A largely follows ideas from standard normal surface theory as described by Matveev [21, §4]. Split handle structures generalise handle structures and allow us to effectively study collections of disjoint normal surfaces with boundary, as seen in the following result. A normal surface in a split handle structure is *duplicate* (see Definition 4.7) if two of its components are normally isotopic or if one is isotopic into the forbidden region.

Definition 1.4. The *size* of a normal surface F, $s(F)$, is the total number of elementary discs in F.

Corollary 4.20. *There exists a constant c_S, which we can take to be $10^{10^{30}}$, such that the following holds. Let M be a manifold with a subtetrahedral split handle structure $\mathcal{H}$. Let $\{\Sigma_i\}$ be a collection of n disjoint normal surfaces in M such that, if we set $\mathcal{H}_0 := \mathcal{H}$ and $\mathcal{H}_i := \mathcal{H}_{i-1} \backslash\backslash \Sigma_i$, then Σ_{i+1} is a non-duplicate fundamental normal surface in $\mathcal{H}_i$. Then there is a normal surface representative of the collection in $\mathcal{H}$ whose size is at most $c_S^{|\mathcal{H}|^2}$.*

The main challenge is to prove that recognition of circle bundles over surfaces with boundary is in **NP**. This is as, by previous work of the author [11], all singular fibres other than those of multiplicity two can be made simplicial in the 82^{nd} barycentric subdivision of any triangulation $\mathcal{T}$ of M, so we can tackle them by drilling them out and then comparing the slope of a meridian of the singular fibre to coordinates on the remaining manifold.

We show in Section 3 that there is a fundamental horizontal surface of minimal degree in almost all Seifert fibered spaces with non-empty boundary. We set up normal surface theory in split handle structures in Section 4. We then use this in Section 5 to show that there is a collection of normal annuli in M which we can usually take to be vertical and which are of at most exponential weight in $|\mathcal{T}|^2$, that cut M into a collection of solid tori. We use these results to prove in Section 6 that recognising circle bundles over surfaces with boundary is in **NP** and giving the surface is in **FNP**. In the first part of Section 7 we extend to the case when M has singular fibres of multiplicity two, and then can use the barycentric subdivision approach described above for the general case.

The recognition problem was previously known to be in **NP** for the following classes of 3-manifolds: the 3-sphere [28], the solid torus, the 3-ball, $S^1 \times S^2$ and $\mathbb{R}P^3$ [10], $\Sigma \times I$ and $\Sigma \tilde{\times} I$ where Σ is a surface [6], and elliptic manifolds [18].

I thank Saul Schleimer for pointing out the connection with torus knot recognition.

2. Background and conventions

All 3-manifolds in this paper are compact and orientable.

Convention 2.1. We take the definition of a handle structure to require the following:

1. each k-handle, with product structure $D^k \times D^{3-k}$, intersects the handles of lower index in exactly $\partial D^k \times D^{3-k}$, and is disjoint from the other k-handles;
2. 1-handles and 2-handles intersect in a manner compatible with their respective product structures; that is, a 1-handle $D^1 \times D^2$ intersects each 2-handle $D^2 \times D^1$ in segments of the form $D^1 \times \gamma$ in the 1-handle and $\lambda \times D^1$ in the 2-handle, where γ and λ are collections of arcs in ∂D^2 in the respective product structures.

Definition 2.2. The *boundary graph* of a 0-handle H is the graph embedded in $\partial H \cong S^2$ whose 0-cells are the intersections with the 1-handles and whose 1-cells are the intersections with the 2-handles.

Definition 2.3. Let $\mathcal{T}$ be a triangulation (or cell structure) of a 3-manifold M. The *dual handle structure* $\mathcal{H}$ for M is formed by taking one $(3-k)$-handle for each k-simplex (or cell) of $\mathcal{T}$ that is not contained in the boundary and gluing them in the corresponding way.

Definition 2.4. A handle structure is *subtetrahedral* if its boundary graph is a subgraph of the complete graph on four vertices; that is, of the boundary graph of the 0-handle that is dual to a tetrahedron disjoint from the boundary of a 3-manifold.

Lemma 2.5. *The dual handle structure to a triangulation of a 3-manifold (possibly with boundary) is subtetrahedral.*

3. Existence of a minimal degree fundamental horizontal surface

Let M be a Seifert fibered space whose boundary is non-empty, equipped with a triangulation $\mathcal{T}$. We will show that, so long as M is not on a short list of exceptions, there is a fundamental horizontal surface in M whose induced covering of the base orbifold of the Seifert fibration is of minimal degree. To do this we use normal surface theory, which was originally developed by Haken [5]. For an exposition of this theory see §3 and §4 of [21]. A normal surface F in $\mathcal{T}$ is *minimal* if it minimises the number of intersections with the 1-skeleton of $\mathcal{T}$ within its isotopy class.

Theorem 3.1 (Theorem 6.5 [12]). *Let F be an orientable, incompressible and ∂-incompressible connected minimal normal surface in an orientable, irreducible and ∂-irreducible manifold M with a triangulation $\mathcal{T}$. Suppose that $nF = G_1 + G_2$ for some n. Then G_1 and G_2 are incompressible and ∂-incompressible, and neither has any components of positive Euler characteristic.*

Jaco and Tollefson's version of this result does not mention the sphere or $\mathbb{R}P^2$ exclusion. However, in Lemma 6.6 of [12] they show that under the same assumptions as in Theorem 3.1, $G_1 \cup G_2$ contains no disc patches, so neither G_1 nor G_2 can be a sphere or $\mathbb{R}P^2$.

In Matveev's book, he gives a variant of this result when F is nonorientable.

Theorem 3.2 (Theorem 4.1.36 [21]). *Let F be an incompressible, ∂-incompressible, minimal connected normal surface F in an orientable, irreducible, ∂-irreducible manifold M with triangulation $\mathcal{T}$, such that $F = G_1 + G_2$. Then G_1 and G_2 are incompressible and ∂-incompressible, and have no components of positive Euler characteristic.*

To produce the desired fundamental surface we will use these theorems and the fact that incompressible surfaces in Seifert fibered spaces are, as one can see from the next few results, very well understood.

Proposition 3.3 ([23, 25]). *An incompressible surface in $T^2 \times I$ is isotopic to one of the following:*

(1) a trivial sphere or disc;
(2) an annulus $\gamma \times I$;
(3) a ∂-parallel annulus or torus;
(4) a nonorientable surface F, which is ∂-compressible and uniquely determined by two different slopes $\frac{p_0}{q_0} = F \cap (T^2 \times 0)$ and $\frac{p_1}{q_1} = F \cap (T^2 \times 1)$ where the curves representing these slopes intersect an even number of times.

In the last case, F has non-orientable genus equal to the length of the minimal sequence of curves in the torus $(\gamma_1, \ldots, \gamma_n)$ from $\frac{p_0}{q_0}$ to $\frac{p_1}{q_1}$ where γ_i and γ_{i+1} have a geometric intersection number of two.

Proposition 3.4 ([23, 24]). *The incompressible, non-∂-parallel, non-S^2 surfaces up to isotopy in a solid torus are classified by their intersections with the boundary, which are as single curves of slopes $\frac{p}{q}$ where q is even (that is, the slopes intersecting a meridian curve an even number of times). Any such surface with non-zero genus is non-orientable and ∂-compressible. Incompressible ∂-parallel surfaces are precisely the annuli which are unions of fibres in fibrations of the solid torus as an S^1-bundle, as well as ∂-parallel discs.*

Definition 3.5. Let M be a Seifert fibered space and let T be a collection of one solid torus neighbourhood of each singular fibre such that T is a union of fibres. A surface in a Seifert fibered space is *pseudo-vertical* if it is isotopic to a surface that is a union of fibres in $M - T$ and is incompressible in each solid torus component of T.

A vertical surface (that is, a union of regular fibres of M) is also pseudo-vertical as we can isotope it to be disjoint from T.

Lemma 3.6. *Let M be an irreducible Seifert fibered space with non-empty boundary, and with (possibly empty) boundary pattern Γ consisting of a collection of vertical fibres. If S is an incompressible, ∂-incompressible surface in M disjoint from the boundary pattern then S is isotopic to a horizontal or pseudo-vertical surface, or is a ∂-parallel disc or a trivial sphere.*

This result is a modification of the standard proof in the case when S is orientable (see [8, Proposition 1.11] or [20, Proposition 10.4.9]) and Mijatović's result in the case where there are no singular fibres [22, Proposition 2.6]). If M is closed then S may additionally be *pseudo-horizontal*; this case is discussed in [4, Theorem 2.5].

Proof. Suppose that S is not a ∂-parallel disc or a trivial sphere. If M is the solid torus, the result follows from Proposition 3.4, noting that S may be ∂-parallel if it is parallel to an annulus in the boundary containing curves of Γ. Otherwise M is ∂-irreducible. Take a collection of disjoint vertical annuli A disjoint from Γ: n that separate a neighbourhood of each singular fibre (that is disjoint from Γ) from M and then some more annuli that cut the remaining part of M into a solid torus. Isotope S such that S is transverse to A and such that $|S \cap A|$ is minimised. Consider $S \cap A$, which consists of arcs and closed curves. Note that $S \cap A$ does not contain any curves that are trivial in S or A, as then by the irreducibility of M and the incompressibility of S we could reduce $|S \cap A|$. It also does not contain any arcs that are ∂-parallel in S or A as S is ∂-incompressible and M is irreducible and ∂-irreducible. As any arc in an annulus that starts and ends on the same boundary component is ∂-parallel, this means that $S \cap A$ contains none of these. Thus S intersects A in a collection of spanning arcs and vertical fibres. By the same reasoning, the same holds for the intersection of S with each annulus of $\partial M - \partial A$. Note that as S is embedded it must intersect each annulus of A and $\partial M - \partial A$ in only one of these two types. If S intersects any annulus in a spanning arc, it intersects all neighbouring annuli to that one in a spanning arc, and hence (as M is connected) all annuli in the collection. Thus the two types are incompatible, so $S \cap A$ must consist of only one of the two.

Let M_0 be $M\backslash\backslash A$, and let S_0 be $S\backslash\backslash A$ in M_0. Let M_1 be the non-singular-fibre-neighbourhood component of M_0, and let S_1 be $S_0 \cap M_1$. We claim that S_0 is incompressible: consider the boundary of some compressing disc. This curve bounds a disc in S as S is incompressible, and that disc intersects the annuli A in simple closed curves. We can use the irreducibility of M to isotope S through this disc, in the process reducing $|S \cap A|$.

Suppose that $S \cap A$ consists of vertical fibres. Recall the classification of incompressible surfaces in the solid torus from Proposition 3.4. Then S_1 is a collection of vertical annuli: it is incompressible, not a meridian disc, and cannot be an incompressible non-orientable surface as one f its boundary curves intersects the meridian once, which is odd. Then the remaining part of S, its intersection with the singular fibre neighbourhoods, can vary: if a given fibre has odd multiplicity, S does not intersect the fibre so must be disjoint from the neighbourhood by the minimality of $|S \cap A|$. If the multiplicity is even, it may intersect the neighbourhood in a punctured non-orientable incompressible surface. Gluing up, we find that S is pseudo-vertical as claimed.

Suppose that $S \cap A$ consists of spanning arcs. (Note that in this case Γ must be empty, as otherwise S would intersect it.) We claim that we can isotope S so that S_0 is ∂-incompressible: if it is not, let D be a non-trivial ∂-compression disc for S_0. Consider the arc $\alpha = \partial D \cap \partial M_0$. We will isotope α so that it is contained in $\partial M_0 \cap \partial M$. In this situation we are done: as S is ∂-incompressible, we can use this boundary compression to reduce $|S \cap A|$. Note that α consists (up to isotopy) of a collection of arcs in annuli. These annuli are alternately from $\partial M_0 \cap \partial M$ and $\partial M_0 \cap A$. If α is contained in one component of $\partial M_0 \cap A$, we can use an isotopy in a collar of the boundary to push α into an adjacent component of $\partial M_0 \cap \partial M$. Otherwise, starting at one end of α, use an isotopy in the collar of the boundary to push this arc α into the adjacent annulus. We can continue this until α is contained in a single annulus. Thus S_0 is ∂-incompressible so is a collection of meridian discs in each of the solid tori, and hence is horizontal. □

We return to finding a fundamental horizontal surface.

Proposition 3.7. *Suppose that M is a Seifert fibered space with non-empty boundary that is not $S^1 \times D^2$, $T^2 \times I$, or $K \tilde{\times} I$. Let $\mathcal{T}$ be a triangulation of M. Let p be the lowest common multiple of the multiplicities of the singular fibres. There is a fundamental horizontal normal surface in M that is a degree p cover of the underlying orbifold. If M is $S^1 \times D^2$, then there is a fundamental normal meridian disc.*

Proof. The solid torus case follows from Corollary 6.4 of Jaco and Tollefson's paper [12]. Otherwise, let n be the number of singular fibres (which may be zero). Note that M is irreducible and ∂-irreducible.

Consider M to be constructed by taking a circle bundle M' over a surface Σ with non-empty boundary, then gluing n solid tori (that is, neighbourhoods of singular fibres) on along vertical annuli with respect to the fibration of M', on a single boundary component of M'. The meridian of one of these solid tori, containing a (p_i, q_i) singular fibre, intersects the gluing annulus in p_i spanning arcs.

Take a degree p horizontal surface in M': if Σ is orientable, this will be p copies of Σ; otherwise it will be $\lfloor \frac{p}{2} \rfloor$ copies of the double cover of Σ, in addition to one copy of Σ if p is odd. Either way, its intersection with each vertical annulus in the boundary of M' will be as p spanning arcs. We can thus take $\frac{p}{p_i}$ meridian discs in each of these singular fibre neighbourhoods and attach them to the degree p horizontal surface in M' to form a connected degree p horizontal surface in M. Now, this surface is incompressible (as it is a finite cover of the base orbifold and hence is π_1-injective) and ∂-incompressible (by the same argument on the double of M), and does not contain any trivial spheres or discs, so let F be a minimal normal surface that is isotopic to it.

Claim 1: The Euler characteristic of F is negative.

Proof: Note that $\chi(F) = p\chi(\Sigma) - \sum_{i=1}^{n} \frac{p}{p_i}(p_i - 1)$, where the sum is over the singular fibres, as adding each meridian disc corresponds to gluing a disc to the surface in M' along p_i different segments of its boundary. As F has boundary,

$\chi(F)$ is positive only if the base surface of M is a disc and M has at most one singular fibre, but in this case $M \cong S^1 \times D^2$. The Euler characteristic is zero if either M is a circle bundle over a surface of zero Euler characteristic, so M is $A \times S^1 \cong T^2 \times I$ or $S \tilde{\times} S^1 \cong K \tilde{\times} I$, or the base orbifold is a disc with two singular fibres, which must both be multiplicity two. In this last case, $M \cong K \tilde{\times} I$. As we have ruled out all of these possibilities in the statement of the proposition, $\chi(F)$ is negative. ■

Suppose that $F = G_1 + G_2$ is a non-trivial sum of normal surfaces that minimises $|G_1 \cap G_2|$ among all such non-trivial decompositions of F. If one of G_1 or G_2 were not connected, we could write G_1, say, as $G_1' \cup G_1''$, and then $F = G_1' + (G_1'' + G_2)$ would be a sum with $|G_1' \cap (G_1'' + G_2)| < |G_1 \cap G_2|$, so the G_i must be connected.

Claim 2: At least one of G_1 and G_2 is horizontal.

Proof: As F is horizontal it is incompressible and ∂-incompressible and is not a trivial disc or sphere. By Theorem 3.2, the same holds for G_1 and G_2. Thus G_1 and G_2 are horizontal or pseudo-vertical by Lemma 3.6.

Suppose both G_1 and G_2 are pseudo-vertical, so are either annuli or are nonorientable surfaces with one boundary component. Note that as ∂F is not a vertical curve, there must be at least one component of each of ∂G_1 and ∂G_2 on each boundary component of M.

Consider the case when p is odd. As summing normal surfaces is additive on homology with $\mathbb{Z}_2$ coefficients, $\partial F = \partial G_1 + \partial G_2$ in $H_1(\partial M; \mathbb{Z}_2)$. Now as F intersects each regular fibre p times, ∂F intersects any boundary component as p' curves where p' divides p and thus is odd. As a consequence, ∂F is nontrivial in the restriction to the $\mathbb{Z}_2$-homology of each boundary component. Since p is odd, there are no even multiplicity singular fibres, so there are no nonorientable pseudo-vertical surfaces. In this case, as G_1 and G_2 are both vertical annuli, we can see that $\partial(G_1 + G_2)$ is trivial in $\mathbb{Z}_2$-homology on at least one boundary component. If there is only one boundary component, $\partial(G_1 + G_2) \equiv 2\partial G_1 \equiv (0, 0) \in H_1(T^2; \mathbb{Z}_2)$. If there are two, as there is at least one component of the boundary of each of the surfaces G_1 and G_2 on each boundary component, $G_1 \cup G_2$ intersects each boundary component in the union of two vertical curves, which similarly is trivial in $\mathbb{Z}_2$-homology. Either way, we have a contradiction.

Otherwise p is even so F is orientable. Then $2F$, the double of F as a normal surface vector, is minimal in its isotopy class and is incompressible and ∂-incompressible as it is just two copies of F. As $2F = 2G_1 + 2G_2$, by Theorem 3.1 the doubles of G_1 and G_2 are incompressible and ∂-incompressible. As $2G_i$ is orientable, and $\partial(2G_i)$ is two copies of ∂G_i and so is a vertical fibre, each G_i is a vertical surface and so has Euler characteristic 0. But as $\chi(2G_i) = 2\chi(G_i)$, each G_i has zero Euler characteristic, so $\chi(F) = 0$, which contradicts Claim 1. ■

We can thus assume that G_1 is horizontal. It remains to show that the degree of its induced covering of the base orbifold is p. Now, G_1 intersects the singular fibre neighbourhoods (which were cut out by vertical annuli) in a collection of meridian discs. A meridian disc around a multiplicity p_i fibre intersects the relevant vertical annulus in p_i spanning arcs. We know that up to isotopy G_1 intersects the circle bundle M' as a horizontal surface, and so in particular intersects each vertical annulus in the boundary component of M' along which we glued the singular fibres the same number of times. Thus this number must be a multiple of all of the multiplicities: that is, it is kp for some integer k, recalling that p is the lowest common multiple of the p_i, and $G_1|_{M'}$ is a degree kp cover of Σ. If $n = 0$, taking the lowest common multiple of the empty set to be 1 by definition, this reasoning holds vacuously.

Note that k is at least one, and $\frac{1}{k}\chi(G_1) = \chi(F) < 0$ by Claim 1. Thus $\chi(G_1)$ is uniquely maximised when $k = 1$; as $\chi(G_1) \geq \chi(F)$, and this maximum achieves equality, k must be 1 and hence G_1 is a degree p horizontal surface. Thus there is a fundamental such surface. $\square$

4. Split handle structures

In this section we introduce split handle structures, which naturally arise when we cut handle structures along normal surfaces. We will use them in Section 5 to show that there is a maximal collection of normal vertical annuli of bounded weight. We keep track of *parallelity pieces*, which occur when these normal surfaces run close to each other, and also of the image of these surfaces in the boundary, which we term the *forbidden region* as later normal surfaces are forbidden to intersect it. We will use the term "sutures" for the boundary of this forbidden region as split handle structures are reminiscent of sutured handle structures, which were devised by Lackenby (see §5 of [17] for an exposition) to allow for normal-surface-type arguments in the context of Scharlemann's combinatorial approach to Gabai's sutured manifold decompositions [27]. Both sutured handle structures and split handle structures act as bookkeeping to record surfaces that have been cut along. We note that normal surfaces in split handle structures are also evocative of the normal surface theory of handle structures with boundary pattern if we require that elementary discs do not intersect the pattern. However, in the boundary pattern case, we usually require that the pattern is contained in the 1-skeleton of the induced handle structure on the boundary (see [21, §3]). Here, the sutures are normal curves.

Definition 4.1. Let P be an I-bundle over a surface Σ such that Σ has non-empty boundary, so P is equipped with a homeomorphism to $\Sigma \overset{(\sim)}{\times} I$. The *horizontal boundary* of P, $\partial_h P$, is $\Sigma \overset{(\sim)}{\times} \partial I$, and the *vertical boundary* of P, $\partial_v P$, is $\partial\Sigma \overset{(\sim)}{\times} I$.

Definition 4.2. A *split handle structure* $\mathcal{H}$ for a compact orientable 3-manifold M is a partition of M into:

(1) k-handles for k between 0 and 3, where each k-handle has a homeomorphism to $D^k \times D^{3-k}$, and
(2) parallelity pieces, each with a homeomorphism to $\Sigma \overset{(\sim)}{\times} I$ for Σ a compact surface

and with a distinguished *forbidden region* $\mathcal{I} \subseteq \partial M$ such that the following conditions hold. Write $\mathcal{H}^k$ for the collection of k-handles, $\mathcal{H}^{\mathcal{P}}$ for the collection of parallelity pieces, and $\partial_h \mathcal{H}^{\mathcal{P}}$ or $\partial_v \mathcal{H}^{\mathcal{P}}$ respectively for the collection of the horizontal or vertical boundaries of the parallelity pieces. The *boundary graph* of a 0-handle H in a split handle structure is the decorated graph in $\partial H \cong S^2$ whose vertices, which we call *islands*, are the components of $H \cap \mathcal{H}^1$, whose edges (which we call *bridges*) are the components of $H \cap \mathcal{H}^2$ and $H \cap \mathcal{H}^{\mathcal{P}}$, and which may have *sutures*, which are the arcs $H \cap \partial\mathcal{I}$. We say that the boundary graph divides ∂H into islands, bridges, *lakes* (components of intersection between ∂H and $\mathcal{H}^3 \cup \partial M - \mathcal{I}$), and *forbidden regions* which are components of $H \cap \mathcal{I}$.

We require that:

(1) each k-handle $D^k \times D^{3-k}$ intersects handles of lower index in exactly $\partial D^k \times D^{3-k}$, and is disjoint from the other k-handles;
(2) the boundary graph of each 0-handle is connected;
(3) each parallelity piece is disjoint from the 2- and 3-handles and the other parallelity pieces;
(4) the forbidden region $\mathcal{I} \subseteq \partial M$ contains $\partial_h \mathcal{H}^{\mathcal{P}}$;
(5) each 1-handle $D^1 \times D^2$ intersects 2-handles $D^2 \times D^1$ in components that are of the form $D^1 \times \gamma$ in the 1-handle and $\lambda \times D^1$ in the 2-handle, where γ and λ collections of arcs in ∂D^2 in the respective product structures;
(6) the intersection of any component $P \cong \Sigma \overset{(\sim)}{\times} I$ of $\mathcal{H}^{\mathcal{P}}$ with a 1-handle $D^1 \times D^2$ is as $D^1 \times \gamma$ in the 1-handle and $\lambda \times I$ in the parallelity piece, where γ is a collection of arcs in ∂D^2 and λ is a collection of arcs in $\partial \Sigma$.

If $\mathcal{I}$ is not empty, we require the following. Write $(\partial\mathcal{H})^k$ for each k for the components of $\partial M \cap \mathcal{H}^k$, and $(\partial\mathcal{H})^{\mathcal{P}}$ for the components of intersection of ∂M with $\mathcal{H}^{\mathcal{P}}$. Note that $(\partial\mathcal{H})^0$ and $(\partial\mathcal{H})^2$ are collections of discs, and $(\partial\mathcal{H})^1$ and $(\partial\mathcal{H})^{\mathcal{P}}$ are collections of discs and possibly some annuli. We require that $\partial\mathcal{I}$ avoids $(\partial\mathcal{H})^2$, runs through discs of $(\partial\mathcal{H})^0$ and $(\partial\mathcal{H})^1$ in arcs that each do not start and end on the same component of $(\partial\mathcal{H})^0 \cap (\partial\mathcal{H})^1$, and intersects each component of $\partial_v \mathcal{H}^{\mathcal{P}} \cap \partial M$ in exactly two arcs or curves, each of which is transverse to the I-bundle structure from the parallelity pieces.

If the forbidden region is empty (which implies that there are no parallelity pieces), this is the usual notion of handle structure.

Definition 4.3. A surface in a split handle structure is *∂-compressible* if it admits a non-trivial ∂-compression disc that is disjoint from the forbidden region.

Whenever we refer to a ∂-compression disc, we require that the disc is disjoint from the forbidden region.

4.1. **Normal surfaces.**

Definition 4.4. A properly-embedded surface in M is *standard* with respect to a split handle structure if it satisfies the following conditions:
(1) it is disjoint from the 3-handles and from the forbidden region;
(2) it is transverse to the I-bundle structure of the 2-handles $D^2 \times I$ and the parallelity pieces $\Sigma \overset{(\sim)}{\times} I$, and is disjoint from their horizontal boundaries;
(3) no component of it is contained in a parallelity piece;
(4) it intersects the 1-handles $D^1 \times D^2$ in a union of D^1-fibres, where each component of this intersection is a disc;
(5) it intersects the 0-handles in discs.

Note that it follows from the definition that if F is a standard surface, then it intersects each 2-handle $D^2 \times D^1$ in sheets of the form $D^2 \times \{*\}$, and similarly intersects each parallelity piece in sheets that are a section of the I-bundle or, if Σ is nonorientable, the double cover of a section.

Definition 4.5. A standard surface F in M with respect to a split handle structure $\mathcal{H}$ is *normal* if it satisfies the following additional conditions:
(1) F intersects each 1-handle $D^1 \times D^2$ in $D^1 \times \lambda$ where λ is a collection of disjoint proper arcs in the island $\{0\} \times D^2$, such that no component of λ

starts and ends on the same connected component of the intersection of the island with a lake;

(2) the intersection of F with each lake does not contain any closed curves or arcs that start and end on the same component of the intersection of the lake with an island;

(3) F intersects each 0-handle in discs, called *elementary discs*, such that the boundary curve of each of these discs crosses each bridge and lake at most once, and if a bridge and a lake are adjacent, intersects only one of the pair.

This is a generalisation of the usual notion of a normal surface in a handle structure. Note that the boundary of an elementary disc of F in the boundary graph of H determines the disc.

Definition 4.6. An *admissible isotopy* of a surface with respect to a split handle structure is an isotopy of the surface in the manifold that fixes the forbidden region $\mathcal{I}$ as a set.

A *normal isotopy* of a normal surface in a split handle structure is an isotopy of the surface in the manifold that fixes each of $\mathcal{I}$, $\mathcal{H}^k$ for each k, and $\mathcal{H}^{\mathcal{P}}$ as a set.

Definition 4.7. A normal surface in a split handle structure with forbidden region $\mathcal{I}$ is *duplicate* if it has two components that are normally isotopic, or it has a component F such that there is a connected component I of the forbidden region, with a collar $C(I) \cong I \times [0,1]$, such that F is normally isotopic to $I \times \{1\}$ in $C(I)$.

The motivation for split handle structures is that their complexity does not grow fast when we cut along a normal surface. We give the normalisation procedure in the appendix (Procedure A.2) and show that it terminates, and in Proposition A.6 prove that if a surface F is incompressible and ∂-incompressible in an irreducible and ∂-irreducible 3-manifold M with a split handle structure $\mathcal{H}$ such that F is disjoint from the forbidden region and no component of F is a sphere or disc or entirely contained in a parallelity piece, then F is isotopic to a normal surface.

Definition 4.8 (Induced split handle structure construction)**.** Let F be a normal surface in M, with respect to a split handle structure $\mathcal{H}$. The *induced split handle structure* on $M\backslash\backslash F$ is constructed as follows.

Consider $\mathcal{H}\backslash\backslash F$. Set its forbidden region $\mathcal{I}$ to be the union of the forbidden region from M and the image of F in $M\backslash\backslash F$. (Note that, as F was normal, these are disjoint.) As F is disjoint from the 3-handles, we can continue to view them as 3-handles in $\mathcal{H}\backslash\backslash F$.

A component of a k-handle in $\mathcal{H}\backslash\backslash F$ will become either a k-handle or a parallelity piece. This is determined as follows. First, if $P \cong \Sigma \overset{(\sim)}{\times} I$ is a parallelity piece of $\mathcal{H}$, as F intersects P in sheets transverse to the I-bundle structure, each component of $P\backslash\backslash F$ inherits an I-bundle structure as either $\Sigma \overset{(\sim)}{\times} I$ or possibly, if Σ is not orientable, as $\tilde{\Sigma} \times I$ where $\tilde{\Sigma}$ is the double cover of Σ. Thus we can view each component of $P\backslash\backslash F$ as a parallelity piece.

If $H \cong D^2 \times D^1$ is a 2-handle of $\mathcal{H}$, a component C of $H\backslash\backslash F$ is itself an I-bundle over D^2, with two horizontal boundary components, each of which arises from intersection with a 3-handle, with $\mathcal{I}$, or with $\partial M - \mathcal{I}$. If both components arise from intersection with $\mathcal{I}$, set C to be a parallelity piece; otherwise, view C as a 2-handle.

If $H \cong D^1 \times D^2$ is a 1-handle of $\mathcal{H}$, a component C of $H\backslash\backslash F$ is a parallelity piece if its boundary consists of the following components: first, two components in the forbidden region $\mathcal{I}$ (whether from the forbidden region in M or arising from intersection with F); second, two components arising from intersection with pieces of 0-handles from M; and finally, the two remaining components where each is a component of intersection with one of ∂M, a single 2-handle, or a single parallelity piece. In this case, C has an I-bundle structure by setting $\partial_h C$ to be the two components in the forbidden region, and choosing a product structure on the remaining boundary, $\partial_v C$, such that each of the four components described above is a union of fibres, and then interpolating. (Note that this product structure can be chosen to be compatible with the product structure on any parallelity pieces defined thus far that C intersects).

Finally, if $H \cong D^0 \times D^3$ is a 0-handle and C is a component of it, consider the boundary of C. Set C to be a parallelity piece if its boundary contains two components of intersection with $\mathcal{I}$ and if there is a product structure $D^2 \times I$ on C such that its intersection with $\mathcal{I}$ is $D^2 \times \partial I$, and each component of its intersection with any of the other handles created so far is of the form $\alpha \times I$, where α is an arc or curve in ∂D^2. Again, note that we can choose this product structure to be compatible with any parallelity pieces that C intersects. Otherwise, set C to be a 0-handle.

Now, take the parallelity pieces of $\mathcal{H}$ to be the union of the parallelity pieces described so far, which we equipped with compatible product structures where they intersected.

We write $\mathcal{H}\backslash\backslash F$ for the induced split handle structure on $M\backslash\backslash F$.

Lemma 4.9. *If F is a normal surface in a split handle structure $\mathcal{H}$, then the induced split handle structure $\mathcal{H}\backslash\backslash F$ is itself a split handle structure.*

Proof. If a piece from a k-handle in $\mathcal{H}\backslash\backslash F$ becomes a parallelity piece in the induced split handle structure, then any pieces from handles of higher index will also be parallelity pieces. As a result, each k-handle $D^k \times D^{3-k}$ will intersect handles of lower index in $\partial D^k \times D^{3-k}$ and will be disjoint from the other k-handles. The boundary graph of each 0-handle will be connected as the boundary graph of each 0-handle arises from taking a connected boundary graph, removing some discs from it, and adding in the boundary of these disc (as we add a new suture for each time the boundary of one of the discs runs through a lake). As F does not intersect any 3-handles, so each one becomes a 3-handle in $\mathcal{H}\backslash\backslash F$, each parallelity piece will be disjoint from the 2- and 3-handles. As intersections of parallelity pieces with 1-handles must necessarily arise from intersections between a 2-handle and a 1-handle, and the product structure of the 2-handle is compatible with that of the parallelity piece, the intersections between parallelity pieces and 1-handles are of the required form. The intersection of $\partial\mathcal{I}$ with the boundary has the required form as F is normal. □

Definition 4.10. A 0-handle H is *semitetrahedral* if it is disjoint from the forbidden region and its boundary graph is a connected subgraph of the complete graph on four vertices. A split handle structure is *semitetrahedral* if all of its 0-handles are.

A 0-handle H is *subtetrahedral* if there is some 0-handle H' in some semitetrahedral handle structure $\mathcal{H}$ with normal surface F such that H is homeomorphic to one of the non-parallelity pieces obtained from H' in the induced split handle structure $\mathcal{H}\backslash\backslash F$.

We say that a split handle structure is *subtetrahedral* if all of its 0-handles are subtetrahedral.

Lemma 4.11. *Let $\mathcal{T}$ be a (material) triangulation of a (compact) 3-manifold M, such that the intersection of each tetrahedron with ∂M is connected and contractible. The dual handle structure to $\mathcal{T}$, formed by taking a $(3-k)$-handle for each k-simplex of $\mathcal{T}$ that is not in $\partial\mathcal{T}$, satisfies the definition of a subtetrahedral split handle structure with empty forbidden region and no parallelity pieces.*

If $\mathcal{H}$ is a subtetrahedral split handle structure, and F is a normal surface in $\mathcal{H}$, then the induced split handle structure on $\mathcal{H}\backslash\backslash F$ is also subtetrahedral.

Proof. The complement of the boundary graph of a 0-handle will deformation retract to the intersection of the corresponding tetrahedron with the boundary, so the handles of this dual handle structure have connected boundary graphs. The second part follows by definition. □

As in the triangulation case, we can view normal surfaces algebraically. Let $\mathcal{H}$ be a subtetrahedral split handle structure. Let d_H be the maximal number of types of elementary disc in any subtetrahedral 0-handle, which is finite by Lemma A.11. To a normal surface F in $\mathcal{H}$ we can associate a vector in $\mathbb{Z}^{d_H|\mathcal{H}|}$ by counting the number of elementary discs in F of each type. Since F must avoid any parallelity piece of $\mathcal{H}$ that does not intersect at least one 0-handle, this vector uniquely determines F. We can interpret summation in $\mathbb{Z}^{d_h|\mathcal{H}|}$ geometrically as follows.

Procedure 4.12 (Normal surface sum in split handle structures)**.** Suppose G_1 and G_2 are normal surfaces in a subtetrahedral split handle structure $\mathcal{H}$, such that if we view them as vectors in $\mathbb{Z}^{d_H|\mathcal{H}|}$, $F = G_1 + G_2$ is also a vector representing a normal surface. We can realise G_1 and G_2 so that they are disjoint in the 0-handles and transverse elsewhere. In the 1-handles, we can use the product structure to ensure that any two components of $G_1 \cap \mathcal{H}^1$ and $G_2 \cap \mathcal{H}^1$ intersect in a single arc that is contained in some $\{*\} \times D^2$ in the 1-handle $D^1 \times D^2$. As we have two embedded surfaces, there are no triple points of intersection. Now, in the 2-handles and parallelity pieces, we know that G_1 and G_2 intersect them in sheets transverse to the I-bundle which are determined by their boundaries. Fix a 2-handle or parallelity piece $H \cong \Sigma \overset{(\sim)}{\times} I$. So long as at most one of $G_1 \cap H$ and $G_2 \cap H$ contains a nonorientable piece, we can realise all the components of $G_1 \cap H$ and $G_2 \cap H$ as follows. Let H' be the complement in H of a collar of the boundary of H. Take the correct number of sheets of G_1 and G_2 in H'. Then interpolate in the collar to achieve the required intersection pattern of G_1 and G_2 in $\partial_v H$, such that $G_1 \cap G_2 \cap H$ contains no closed curves. If both of them contain a nonorientable sheet in H, then we can realise the two sheets to intersect in H' in a single arc, and then again interpolate in the collar to achieve the required intersection pattern. Now at each arc of intersection of G_1 and G_2 in H, if we cut along the arc then we have two choices of how to reglue to resolve the intersection: a choice of *switch*. This is determined by the picture at one of the arc's endpoints p in $\partial_v H$. The *regular switch*, which is the choice of gluing that produces sheets transverse to the product structure, produces two sheets of the same types as those we started with (unless both the sheets were nonorientable, in which case it produces one sheet that is a double of one of them).

We wish to show that choosing all regular switches is compatible. As the product structures of the 1-handles and the 2-handles or parallelity pieces are transverse, if we consider the arc of intersection in the 1-handle containing p, the choice of regular switch at p induces the regular switch on the other end of the arc. Similarly, as the regular switch is the choice giving sheets transverse to the product structure on the 2-handle or parallelity piece, if we consider the arc in H containing p, the choice of regular switch at p induces the regular switch along the whole arc. Thus there is a global *regular switch* for $G_1 \cup G_2$, which produces F.

Definition 4.13. The weight of a normal surface F is $(p(F), b(F), |F \cap \partial M|)$, where $p(F)$ is $|F \cap (\partial \mathcal{H}^2 \cup \partial \mathcal{H}^{\mathcal{P}})|$ and $b(F)$ is $|F \cap \mathcal{H}^1|$.

Lemma 4.14. *Let G_1 and G_2 be normal surfaces such that the vector $F = G_1 + G_2$ also corresponds to a normal surface. Let F' be an incompressible and ∂-incompressible surface constructed by resolving the curves of intersection of $G_1 \cup G_2$ that includes at least one irregular switch. Then F' is isotopic to a normal surface of lower weight than F.*

Proof. Take G_1 and G_2 to minimise $|G_1 \cap G_2|$ in their normal isotopy class. The plate degree $p(F)$ is $p(G_1) + p(G_2)$ as the intersection of F with the boundaries of the 0-handles is the union of the intersections of G_1 and G_2 with them. Suppose we take the irregular switch at some point p in the boundary of a 2-handle or parallelity piece H. Let c be the number of essential curves of G_1 and G_2 in the component of $\partial_v H$ containing p. Write F' for the result of this irregular switch at p. The point p is in some component of intersection Q of H with a 1-handle or the boundary, as $G_1 \cup G_2$ is disjoint in the 0-handles.

Suppose that p is in the boundary of a 1-handle or a component of intersection with the boundary that is a disc. This disc Q naturally has a boundary comprised of four edges: two arcs in $\partial_h H$ and two arcs in $\partial_v H$. Note that the two edges of ∂Q in $\partial_v H$ each intersect $G_1 \cup G_2$ in c points. However in the irregular switch case, $Q \cap F'$ now contains at most $c - 2$ spanning arcs, so it contains an arc that starts and ends on the same edge of ∂Q: a *return* in the language of Matveev [21, §4]. But then this gives us a disc of $F' \cap \mathcal{H}^0$ whose boundary runs through the same bridge twice, so F' is not normal. When we normalise F' in Procedure A.2, in Move 2 we will reduce its plate degree by at least two, so F' is isotopic to a normal surface of plate degree at most $p(F) - 2$.

If p is in a component of intersection with the boundary that is an annulus (that is, if the component of $\partial_v H$ containing p is entirely in the boundary) then in most cases – if at most one of the sheets containing p is nonorientable, or if p is one end of an arc of intersection that projects to an inessential arc in Σ – we could have isotoped G_1 and G_2 to remove this intersection. The only case when this is not possible is if both the sheets are nonorientable and p is an endpoint of an essential arc of intersection. If we take the irregular switch at p, then there is a spanning arc of this annulus of $\partial_v H$ that intersects the resulting configuration $c - 2$ times, so again in Move 2 we will reduce the plate degree of F' by at least two. □

Lemma 4.15. *A normal surface in a subtetrahedral split handle structure has non-zero beam degree.*

Proof. As no component of a normal surface is contained in a parallelity piece, every normal surface contains at least one elementary disc. It suffices to show that the boundary of each elementary disc intersects at least one island. Since the boundary graph of each 0-handle is connected and contains at least one island, every curve contained in a lake is trivial and each bridge starts and ends on an island. The boundary of the elementary disc runs through a bridge or lake and thus through some island. □

4.2. **Fundamental normal surfaces.** Fix a subtetrahedral split handle structure $\mathcal{H}$ of a manifold M and recall that we can associate a vector in $\mathbb{Z}^{d_H|\mathcal{H}|}$ to each normal surface in $\mathcal{H}$, where by Remark A.13 d_H is at most $13 \cdot 13!$, so is bounded. A vector v in $\mathbb{Z}^{d_H|\mathcal{H}|}$ corresponds to a normal surface F if and only if it satisfies the following conditions:

(1) each coordinate of v is nonnegative;
(2) fixing a parallelity piece P and a component of intersection C_1 of P with a 0-handle, for any other component C_2 of intersection of P with $\mathcal{H}^0$, $|P \cap C_1| = |P \cap C_2|$;
(3) for each 1-handle $D^1 \times D^2$, which has two components of intersection with 0-handles, for each arc type in $\{0\} \times D^2$ up to normal isotopy, the number of elementary discs in each 0-handle of the types that intersect the 1-handle in that arc type are the same; and
(4) for each pair of elementary disc types in a subtetrahedral 0-handle, if they have an essential intersection then v contains only one of them.

The first three conditions give us a cone C in $\mathbb{Z}^{d_H|\mathcal{H}|}$, as they are linear. Note that if $v \in C$ satisfies the fourth condition and $v = t + u$ for $t, u \in C$, then t and u also satisfy the fourth condition.

Definition 4.16. A normal surface F is *fundamental* if whenever $F = G_1 + G_2$ as a sum of normal surfaces, one of G_1 and G_2 is empty.

Note that a fundamental normal surface must be in any (minimal) set that spans C with $\mathbb{Z}^+$ coefficients: a *minimal Hilbert basis.* Linear integer programming techniques, as used by Haken [5] and Hass-Lagarias-Pippenger [7, §6], tell us that there is a universal constant c such that any fundamental normal surface with respect to a triangulation with t tetrahedra has size at most c^t, where we recall the definition of *size* from Definition 1.4. Their work was in the setting of normal surfaces in triangulations but the general linear programming approach applies generally, as summarised by Lackenby:

Proposition 4.17 (Theorem 8.1 [17])**.** *Suppose that A is a $m \times n$ matrix. Consider the cone of solutions to $Ax = 0$, subject to the constraint that all entries of x are nonnegative. Suppose that each row of A has ℓ^2 norm at most k. If x is a fundamental solution (i.e. is in the integral Hilbert basis) to this system, then each coordinate of x is bounded by $n^{3/2}k^{n-1}$.*

We can apply these bounds in our setting as follows.

Lemma 4.18. *There exists a constant c_F, which we can take to be $2^{74+74\cdot 13\cdot 13!}$, such that if G is a fundamental normal surface in a subtetrahedral split handle structure $\mathcal{H}$ then the size of G is bounded by $c_F^{|\mathcal{H}|}$.*

For this proof, we will use the following result from the appendix on the combinatorics of subtetrahedral split handle structures:

Lemma A.7. *Let H be a subtetrahedral 0-handle. Let G be its boundary graph in $\partial H \cong S^2$. Then G contains between one and four islands; each island has at most three components of intersection with bridges; and if b is the number of bridges of G, then the number of sutures of G at most $12 - 2b$ and an island intersecting v bridges has at most $6 - 2v$ intersections with sutures.*

Proof of Lemma 4.18. Note that n is at most $d_H|\mathcal{H}|$. The constraints from the parallelity piece equations give at most $6|\mathcal{H}|$ equations (one for each bridge), where the form of each equation is to set the sum of two lists of elementary discs to be equal, where each list is contained in one 0-handle, and so its ℓ^2 norm is at most $2d_H$.

As each 0-handle has at most four islands, there are at most $2|\mathcal{H}|$ 1-handles. Fix one of these islands, I. By Lemma A.7 it intersects at most three bridges – let v be this number – and at most $6 - 2v$ sutures. As each part of the forbidden region that intersects I is bounded on each side by sutures or sides of bridges, of which we have a total of at most six, I has at most three components of intersection with the forbidden region. We thus have divided the boundary of I into at most 9 possible segments for elementary discs to enter and leave by (corresponding to intersections with bridges and lakes, but not the forbidden region), where an arc of intersection of an elementary disc with I is determined by its entry and exit segments. There are thus at most $8 \cdot 9 = 72$ arc types in I from elementary discs. Thus each 1-handle gives at most 72 equations, of the same form as in the parallelity piece case: setting some sums of elementary disc types to be equal, which will be a row of A of ℓ^2 norm at most d_H.

The number of elementary discs of any given type of G is thus bounded by $(d_H|\mathcal{H}|)^{3/2} d_H^{d_H|\mathcal{H}|-1}$. We can then find a c_F as follows:

$$\begin{aligned}|G| &\leq |\mathcal{H}|(d_H|\mathcal{H}|)^{3/2} d_H^{d_H|\mathcal{H}|} \\ &= 2^{3/2\log(d_H)+5/2\log(|\mathcal{H}|)+d_H\log(d_H)|\mathcal{H}|} \\ &\leq 2^{2(d_H+1)\log(d_H)|\mathcal{H}|}\end{aligned}$$

so we can take c_F to be $2^{2(d_H+1)\log(d_H)}$ which, as by Remark A.13 d_H is at most $13 \cdot 13! < 2^{37}$, is bounded above by $2^{74+74\cdot 13\cdot 13!}$. □

The following result generalises an idea first developed by Simon King in his doctoral thesis to find a maximal collection of normal spheres in a 3-manifold [14, §3.2 and §3.3].

Lemma 4.19. *There exists a constant c_B, which we can take to be $2^{182\cdot 13!}$, such that the following holds. Let M be a manifold with a subtetrahedral split handle structure $\mathcal{H}$. Let $\{\Sigma_i\}$ be a collection of n disjoint normal surfaces in M such that, if we set $\mathcal{H}_0 := \mathcal{H}$ and $\mathcal{H}_i := \mathcal{H}_{i-1}\backslash\backslash\Sigma_i$, then Σ_{i+1} is non-duplicate and of size at most $2^{f(|\mathcal{H}_i|)}$ in the induced split handle structure on $\mathcal{H}_i$ for some (increasing) function f that does not depend on i and has $f(1) \geq 1$. Then there is a normal surface representative of the collection in $\mathcal{H}$ whose size is at most $c_B^{|\mathcal{H}|f(2^{13}|\mathcal{H}|)}$.*

Proof. By Lemma 4.9 $\mathcal{H}_i$ is indeed a split handle structure for each i. Let $s(i)$ be the size of surfaces $\Sigma_i, \ldots, \Sigma_n$ included into $\mathcal{H}_{i-1}$. As Σ_{i+1} avoids the forbidden region of $\mathcal{H}_i$, it is disjoint from Σ_i, so their normal sum is their disjoint union. Now, when we include the surfaces $\Sigma_{i+1}, \dots, \Sigma_n$ into $\mathcal{H}_{i-1}$ and consider their size, two things happen: some of the parallelity pieces of $\mathcal{H}_i$ may be broken up into many 0-handle pieces and some of the elementary disc types of $\mathcal{H}_i$ may be identified, which does not change the total count of them in the surfaces. As the parallelity pieces had to have been been created from cutting along Σ_i, each sheet of the surfaces in a parallelity piece in $\mathcal{H}_i$ decomposes into at most $|\Sigma_i|$ 0-handle pieces in $\mathcal{H}_{i-1}$. As each sheet is adjacent to some elementary disc in a 0-handle of $\mathcal{H}_i$, and (as there are most six bridges in each of these 0-handles) each elementary disc runs through at most six bridges, the number of sheets is at most $6s(i+1)$. Thus when we include $\Sigma_{i+1}, \dots, \Sigma_n$ into $\mathcal{H}_{i-1}$, the new weight $s(i)$ is at most $|\Sigma_i| + s(i+1) + 6s(i+1)|\Sigma_i| \leq 8s(i+1)|\Sigma_i|$.

By Lemma A.10, $|\mathcal{H}_i|$ is at most $2^{13}|\mathcal{H}|$ in $\mathcal{H}_{i+1}$ for all i. Thus $|\Sigma_i| \leq 2^{f(2^{13}|\mathcal{H}|)}$ for all i, so $s(i) \leq s(n)(8 \cdot 2^{f(2^{13}|\mathcal{H}|)})^{n-i}$. Thus we have that $s(0)$ is at most $8^n 2^{(n+1)f(2^{13}|\mathcal{H}|)}$. By a Kneser-Haken finiteness argument, as Σ_{i+1} is non-duplicate either it and another Σ_i are a pair of a nonorientable surface and its orientable double cover, or it contains some elementary disc that is not in a parallelity region, so n is at most $4d_H|\mathcal{H}|$. Now, we bound c_B as follows:

$$\begin{aligned} s(0) &\leq 2^{12d_H|\mathcal{H}|} 2^{(d_H|\mathcal{H}|+1)f(2^{13}|\mathcal{H}|)} \\ &\leq 2^{14d_H|\mathcal{H}|f(2^{13}|\mathcal{H}|)} \end{aligned}$$

which, as by Remark A.13 d_H is at most $13 \cdot 13!$, gives us the bound. □

Corollary 4.20. *There exists a constant c_S, which we can take to be $10^{10^{30}}$, such that the following holds. Let M be a manifold with a subtetrahedral split handle structure $\mathcal{H}$. Let $\{\Sigma_i\}$ be a collection of n disjoint normal surfaces in M such that, if we set $\mathcal{H}_0 := \mathcal{H}$ and $\mathcal{H}_i := \mathcal{H}_{i-1} \backslash\backslash \Sigma_i$, then Σ_{i+1} is a non-duplicate fundamental normal surface in $\mathcal{H}_i$. Then there is a normal surface representative of the collection in $\mathcal{H}$ whose size is at most $c_S^{|\mathcal{H}|^2}$.*

Proof. This immediately follows from Lemmas 4.18 and 4.19, as well as the fact that $c_B^{2^{13}\log_2 c_F} < 2^{182 \cdot 13! \cdot 2^{13}(74+74 \cdot 13 \cdot 13!)} < 10^{10^{30}}$. □

5. A complete bounded collection of normal vertical annuli

We will use normal surfaces in split handle structures, as developed in Section 4 and Appendix A, to show that there is a maximal collection of normal vertical annuli of bounded weight. The main result of Appendix A is the following.

Proposition A.16. *Let M be an irreducible ∂-irreducible 3-manifold and let $\mathcal{H}$ be a subtetrahedral split handle structure for M. Let F be a normal surface of minimal weight in its (admissible) isotopy class which is incompressible and ∂-incompressible, with $F = G_1 + G_2$. Then G_1 and G_2 are also incompressible and ∂-incompressible, and neither G_1 nor G_2 is a copy of S^2, $\mathbb{R}P^2$ or a disc.*

5.1. Fundamental Möbius bands.

Lemma 5.1. *If S is a Möbius band in an orientable irreducible 3-manifold M where ∂S is non-trivial in $\pi_1(M)$, then the double of S is essential unless M is $S^1 \times D^2$.*

Proof. As the double of S, $\tilde{S}$, is an annulus, there is only one loop (up to isotopy) in $\tilde{S}$ that does not bound a disc in $\tilde{S}$: one of its boundary components. As this is isotopic to ∂S, it is non-trivial in $\pi_1(M)$ so cannot bound a disc in M either, so $\tilde{S}$ is incompressible. As $\tilde{S}$ is the boundary of a neighbourhood of S, it is separating. Since M is irreducible, an annulus is ∂-parallel if and only if it is ∂-compressible. In this case, we can note that the boundary of a neighbourhood of a Möbius band is not, in fact, boundary-parallel in this neighbourhood. So if $\tilde{S}$ is not essential, it cuts M into $S^1 \times I \times I$ and $S \tilde{\times} I$, so M is $S \tilde{\times} I \cong S^1 \times D^2$. □

Lemma 5.2 (Lemma 2.4 [22])**.** *Suppose that M is irreducible, ∂-irreducible, and is not $T^2 \times I$ or $K \tilde{\times} I$. Let S be a toroidal boundary component of M. If A and B are properly-embedded incompressible and ∂-incompressible annuli in M (that are not necessarily disjoint), such that at least one boundary component of each is on S, then these boundary components are isotopic.*

Lemma 5.3. *Suppose that M is a Seifert fibered space with non-empty boundary, not homeomorphic to a solid torus or $K \tilde{\times} I$, and has a subtetrahedral split handle structure $\mathcal{H}$, where the forbidden region (if non-empty) is a collection of vertical annuli. If M has any multiplicity two singular fibres, then for each boundary component T of M, there is a fundamental normal pseudo-vertical Möbius band containing one of these fibres whose boundary is on T.*

Proof. Note that no Seifert fibration of $T^2 \times I$ has multiplicity two singular fibres. Let S be a pseudo-vertical Möbius band which is incompressible, ∂-incompressible, and disjoint from $\mathcal{I}$. Up to isotopy, we can consider it to be a union of regular fibres and the multiplicity two singular fibre. By Lemma 5.1 the double of S, $\tilde{S}$, is essential in M with empty forbidden region, and so is essential in $\mathcal{H}$.

Let F be a minimal normal surface representative of S. Suppose that $F = G_1 + G_2$. By Proposition A.16, G_1 and G_2 are incompressible and ∂-incompressible, and each has Euler characteristic at most zero, and so (as $\chi(F) = 0$) equal to zero. By Lemma 3.6 they are both thus horizontal or (pseudo-)vertical. If one is a Möbius band, say G_1, taking the double and observing by the same reasoning as above that it is essential, we can see by Lemma 5.2 that ∂G_1 is isotopic to ∂F. It is thus a pseudo-vertical Möbius band, so is (up to isotopy) a union of regular fibres and a single multiplicity two singular fibre. Note that as the boundary of G_1 is a summand of that of F, ∂G_1 is contained in T. By Lemma 4.15 the weight of G_2 is non-zero so the weight of G_1 is strictly less than that of F.

As at least one of G_1 and G_2 has boundary and we have dealt with the Möbius band case, either both are annuli or one is an annulus and the other is closed. In either of these cases, we can apply Lemma 5.2 to the boundary components of the essential annuli to see that their boundary curves are isotopic to ∂F. There are an even number of boundary curves among G_1 and G_2, so $\partial G_1 + \partial G_2$ is trivial in $H_1(\partial M; \mathbb{Z}/2)$. But ∂F is not as it is a single curve, and normal sum is additive on homology with $\mathbb{Z}_2$ coefficients, so we have a contradiction. Thus there is a fundamental pseudo-vertical Möbius band. □

5.2. **Fundamental vertical annuli.**

Lemma 5.4. *The only Seifert fibered spaces containing horizontal Möbius bands are those with the Seifert data $[D^2, 1/2, 1/2]$ or that of a circle bundle over a Möbius* *band. The only additional Seifert fibered space containing a horizontal annulus has the Seifert structure of a circle bundle over an annulus.*

Note that in the first two cases M is homeomorphic to $K \tilde{\times} I$.

Proof. Write S for the Möbius band, and let M be such a Seifert fibered space. As $\chi(S) = 0$, the base orbifold G has $\chi(G) = 0$. If we write $\chi(G) = 2-a-b-\sum_i(1-\frac{1}{p_i})$ where b is the number of boundary components, a is twice its genus (if orientable) or its nonorientable genus (otherwise), and $\{p_i\}$ are the multiplicities of the singular fibres, then we can see that $a + (b-1) + \sum_i(1 - \frac{1}{p_i}) = 1$. As $b \geq 1$, each of these terms is nonnegative.

That M contains a horizontal Möbius band is equivalent to the statement that the base orbifold G of M is covered by a Möbius band. We can work by cases: if $a > 0$, we have a Möbius band with no singular fibres. If $b > 1$, then b must be 2 so we have an annulus, which the Möbius band does not cover but the annulus (trivially) does. Otherwise, $a = 0$ and $b = 1$, so G is a disc with some singular fibres with multiplicities such that $\sum_i(1-\frac{1}{p_i}) = 1$. For each $p > 1$, we have $\frac{1}{2} \leq 1-\frac{1}{p} < 1$, so the only solution is two singular fibres, each with multiplicity two. □

Lemma 5.5. *Let F be a connected non-duplicate normal surface in a subtetrahedral split handle structure $\mathcal{H}$. If $F = G_1 + G_2$ then neither G_1 nor G_2 is both connected and duplicate.*

Proof. If G_1 were connected and duplicate, which is to say that there was a normal isotopy taking it to the boundary of a collar of a forbidden region component, then this isotopy would also make it disjoint from G_2, so then $F = G_1 + G_2$ would have two components. □

Lemma 5.6. *Let M be a Seifert fibered space with non-empty boundary, other than the solid torus. Let $\mathcal{H}$ be a subtetrahedral split handle structure for M where the forbidden region $\mathcal{I}$ (if it is non-empty) is a collection of vertical annuli. Then $\mathcal{H}$ contains a fundamental normal surface, disjoint from the forbidden region, that is either an essential Möbius band or an essential non-duplicate annulus. Furthermore, if M is not homeomorphic to $T^2 \times I$ or $K \tilde{\times} I$ or $\mathcal{I}$ is not empty, then this surface is vertical with respect to the Seifert fibered structure.*

Proof. We have already proved this in Lemma 5.3 if M has a multiplicity two singular fibre and is not $K \tilde{\times} I$. If M does have a multiplicity two singular fibre, then, we can assume that it is $K \tilde{\times} I$ with the $[D^2, 1/2, 1/2]$ Seifert fibration.

Let A be an incompressible ∂-incompressible annulus or Möbius band in M that is disjoint from the forbidden region and is not isotopic into it. This exists as M is not the solid torus. By Proposition A.6 A has a normal representative F. As A is incompressible, it cannot be isotoped to be inside a single handle, so we can let F be of minimal weight in the admissible isotopy class of A (that is, up to isotopy fixing $\mathcal{I}$). Suppose that $F = G_1 + G_2$ as a non-trivial sum of normal surfaces, where we choose G_1 and G_2 to minimise $|G_1 \cap G_2|$. By Proposition A.16, G_1 and G_2 are also incompressible and ∂-incompressible, and have Euler characteristic at most 0. If (say) G_1 has multiple connected components, so $F = G_1' + G_1'' + G_2$, then by resolving the sum $G_1'' + G_2$ (which we note does not add any curves of intersection with G_1') we get $F = G_1' + (G_1'' + G_2)$, where $|G_1 \cap G_2| > |G_1' \cap (G_1'' + G_2)|$, so both G_1 and G_2 must be connected. As $\chi(G_1) + \chi(G_2) = \chi(A) = 0$, both have Euler characteristic exactly zero – that is, each component is an annulus, Möbius band, Klein bottle or torus. By Lemma 5.5 neither is duplicate.

If one is closed, the other, say G_1, has the same boundary as A, so is an essential non-duplicate annulus or Möbius band also. If both are essential annuli, then we have two essential non-duplicate annuli of smaller weight than A.

If G_1 is a Möbius band then by Lemma 3.6 it is either horizontal, in which case $\mathcal{I}$ must be empty and M must be $K \tilde{\times} I$, or pseudo-vertical so M is again $K \tilde{\times} I$ with the Seifert structure $[D^2, 1/2, 1/2]$.

In any of these cases, we have produced an incompressible and ∂-incompressible non-duplicate annulus or Möbius band that is of smaller weight than F, so there is a fundamental such surface G. If $\mathcal{I}$ is not empty then G cannot be horizontal, so must be (pseudo-)vertical and, if a Möbius band, have an essential vertical double. Otherwise, if G is horizontal, then by Lemma 5.4 M is homeomorphic to $K \tilde{\times} I$ or $T^2 \times I$. □

We will use Lemmas 4.18 and 4.19 to prove that we can obtain a complete collection of these annuli of bounded size. Recall that by Lemma 4.11, cutting along a normal surface produces another subtetrahedral split handle structure.

Proposition 5.7. *There exists a constant c_A, which we can take to be $10^{10^{30}}$, such that the following holds. Let M be a Seifert fibered space with non-empty boundary, other than the solid torus. Let $\mathcal{H}$ be a subtetrahedral handle structure for M that is dual to a triangulation $\mathcal{T}$ such that the intersection of each tetrahedron of $\mathcal{T}$ with ∂M is connected and contractible. There is a maximal collection of disjoint normal essential annuli in $\mathcal{H}$, so that no two are isotopic, of total size at most $c_A^{|\mathcal{H}|^2}$. If M is not $T^2 \times I$ or $K \tilde{\times} I$, then these annuli are vertical.*

Proof. By Lemma 4.11 $\mathcal{H}$ can be viewed as a subtetrahedral split handle structure with empty forbidden region. Then by Lemma 5.6 there is a fundamental normal surface F in $\mathcal{H}$ that is either an essential Möbius band or an essential non-duplicate annulus. Note that by Lemma 4.11 $\mathcal{H}\backslash\backslash F$ is also a subtetrahedral split handle structure. If M was not $T^2 \times I$ or $K \tilde{\times} I$, then we can take the first annulus to be vertical, so the forbidden region of $\mathcal{H}\backslash\backslash F$ will be a non-empty collection of vertical annuli. We can then continue to apply Lemma 5.6 to generate a collection of disjoint normal incompressible and ∂-incompressible annuli A_i and handle structures $\mathcal{H}_{i+1} = \mathcal{H}_i\backslash\backslash A_i$, where the annulus A_i is fundamental in $\mathcal{H}_i$ or is the double of a fundamental surface. This is almost the situation of Corollary 4.20, save that some surfaces may be doubles of fundamental surfaces, rather than being fundamental themselves. Each annulus (by Lemma 4.18) is thus of size at most $2c_F^{|\mathcal{H}_i|}$ where $c_F = 2^{74+74\cdot 13\cdot 13!}$, so is of size at most $2^{75+74\cdot 13\cdot 13!|\mathcal{H}_i|}$. By Lemma 4.19 then the size of the whole collection is at most $c_B^{(75+74\cdot 13\cdot 13!)|\mathcal{H}|2^{13}|\mathcal{H}|}$ where we can set c_B to be $2^{182\cdot 13!}$. (In particular, the collection is finite.) Thus the total bound is at most $c_A^{|\mathcal{H}|^2}$ where we can set

$$c_A = 2^{182\cdot 13!\cdot 2^{13}\cdot(75+74\cdot 13\cdot 13!)}.$$

This is a maximal collection of non-duplicate vertical essential annuli; there is thus a subset of it that is a maximal collection of non-isotopic vertical essential annuli. One can compute that $c_A < 10^{10^{30}}$.

If M is $T^2 \times I$, then by Proposition 3.3 any essential annulus in $T^2 \times I$ is $\gamma \times I$ for some essential curve γ in T^2, and then the complement of this annulus is a solid torus with two parallel annuli as its forbidden region, so there are no further isotopy classes of essential annuli. If M is $K \tilde{\times} I$, it has a Seifert fibered structure as $S \tilde{\times} S^1$, where S is the Möbius band. An essential annulus is either horizontal or vertical. If horizontal, it separates M into two copies of $S \tilde{\times} I$, which are each homeomorphic to a solid torus with a single annulus as their forbidden region so contain no essential annuli that are not isotopic to the one we already cut along. If vertical, it cuts M into a single solid torus, with two parallel annuli as its forbidden region, so again there are no further annuli. In these cases we took a single fundamental annulus, or double of a fundamental Möbius band, which is of size at most $2c_F^{|\mathcal{H}|}$ for $c_F = 2^{74+74\cdot 13\cdot 13!}$ by Lemma 4.18, which is certainly smaller than bound on c_A we have given. $\square$

Lemma 5.8. *Let $\mathcal{H}$ be a handle structure for an irreducible ∂-irreducible 3-manifold M that is dual to a triangulation $\mathcal{T}$ such that the intersection of each tetrahedron of $\mathcal{T}$ with ∂M is connected and contractible. Let F be a non-duplicate incompressible ∂-incompressible normal surface in $\mathcal{H}$. There is a normal surface F' in $\mathcal{T}$ that is isotopic to F such that the weight of F' (that is, its number of intersections with the 1-skeleton) is at most $16|\mathcal{T}|s(F)$.*

Proof. By Lemma 4.11 $\mathcal{H}$ is a subtetrahedral split handle structure. Each elementary disc of F runs over at most four bridges, so its number of intersections with the 1-skeleton of $\mathcal{T}$ will be at most four. The barrier to directly including F into $\mathcal{T}$ is that an elementary disc in a subtetrahedral handle does not necessarily correspond to one in the dual tetrahedron: the ones that do not are the ones that run over vertices in the boundary of $\mathcal{T}$. It suffices then to perturb F enough that it is transverse to $\mathcal{T}$, without increasing its weight too much, as then, as F is incompressible and ∂-incompressible, we can normalise it which does not increase its weight.

Map F into $\mathcal{T}$ by sending each elementary disc of F to a disc in a tetrahedron of $\mathcal{T}$ that is transverse to the triangulation except that it may run over vertices in $\partial\mathcal{T}$. Each elementary disc of F ran over at most four bridges, so this surface intersects the 1-skeleton of $\mathcal{T}$ in at most $4s(F)$ points. At each vertex in $\partial\mathcal{T}$, consider its link L, which is a disc as M is a manifold. The elementary discs of F intersect L in a set of disjoint arcs. Pick a coherent choice of *direction* transverse to each arc, so that no two arcs point towards each other. At each arc, replace the portion of F that runs through the vertex with the subset of L in the chosen direction. We thus obtain a surface F' isotopic to F and transverse to the triangulation. As L intersects each edge of $\mathcal{T}$ at most twice, this operation adds at most two points of intersection of each elementary disc with each edge of $\mathcal{T}$. There are at most $6|\mathcal{T}|$ edges so the weight of F' is at most $(4 + 12|\mathcal{T}|)s(F)$. Normalising F' produces a surface isotopic to F and of no greater weight than F'. $\square$

Corollary 5.9. *There exists a constant c_T, which we can take to be $10^{10^{36}}$, such that the following holds. Let $\mathcal{T}$ be a triangulation of a Seifert fibered space M with non-empty boundary, other than the solid torus. There is a collection of disjoint normal essential annuli in $\mathcal{T}$, such that their complement is a collection of solid tori, of total weight at most $c_T^{|\mathcal{T}|^2}$. We can take these annuli to be vertical so long as M is not $T^2 \times I$ or $K \tilde{\times} I$.*

Proof. Consider the handle structure $\mathcal{H}$ that is dual to the 1^{st} barycentric subdivision of $\mathcal{T}$, $\mathcal{T}^{(1)}$, which contains $24|\mathcal{T}|$ tetrahedra. In $\mathcal{T}^{(1)}$ the intersection of each tetrahedron with ∂M is connected and contractible. Thus $\mathcal{H}$ satisfies the hypotheses of Proposition 5.7 so there is a maximal collection C of disjoint normal essential annuli in $\mathcal{H}$, such that no two are isotopic, of total size at most $c_A^{|\mathcal{H}|^2}$, and so that if M is not $T^2 \times I$ or $K \tilde{\times} I$ then these annuli are vertical. We can now apply Lemma 5.8 to this collection C to obtain an isotopic set of normal surfaces in $\mathcal{T}^{(2)}$ of weight at most $16 \cdot 24^2|\mathcal{T}|c_A^{24^4|\mathcal{T}|^2} \leq c_A^{2\cdot 24^4|\mathcal{T}|^2}$ where $c_A < 10^{10^{30}}$. We can thus set c_T to be $c_A^{2\cdot 24^4}$, which gives the desired bound.

Now, if M is $T^2 \times I$ or $K \tilde{\times} I$ we took a single annulus that cut M into one or two solid tori, so let this be the minimal collection. Otherwise, all the annuli are vertical. They lie over essential arcs in the base orbifold of M. If there are n singular fibres, there must be n annuli separating neighbourhoods of singulars fibre from the remainder of M. Consider the dual graph to the rest of the collection: it has a vertex for each region of $M \backslash\backslash C$, and an edge for each annulus, connecting the vertices of the regions it bounds. Take the complement of a spanning tree of this dual graph. This collection of annuli will cut the remainder of the orbifold into a single disc, and so will cut the remainder of M into a single solid torus. Take these annuli as well as the ones that cut off singular fibre neighbourhoods as the minimal collection. □

6. Recognising circle bundles over surfaces with boundary is in NP

Lemma 6.1. *Let $\mathcal{T}$ be a triangulation of the solid torus. Then there is a fundamental normal meridian disc in $\mathcal{T}$ and a normal curve in $\partial\mathcal{T}$ of weight at most exponential in $|\mathcal{T}|$ that intersects the disc once.*

Proof. Corollary 6.4 of [12] tells us that if M is ∂-irreducible then there is a vertex normal essential disc, and the only essential disc in $S^1 \times D^2$ is the meridian disc. Take a normal curve γ in $\partial\mathcal{T}$ that intersects this meridian disc once. If $\gamma = \alpha_1 + \alpha_2$ as a normal sum, then one of the α_i must also intersect the disc once, so there is a fundamental normal curve in the triangulation on the boundary that satisfies this property. □

To determine the slope of curves in a torus we will need to compute their algebraic intersections quickly. A normal curve is not equipped with an orientation, so the algebraic intersection does not come with a sign; however, if we have more than two (non-disjoint) curves, then picking a sign for $i(\alpha, \beta)$ and $i(\beta, \gamma)$ determines that of $i(\alpha, \gamma)$. As in [3, §6], we can represent an oriented normal curve by giving its algebraic intersection number with each (oriented) edge of the triangulation. We will use the following result to assign orientations to normal curves and then compute their algebraic intersections. Schaefer, Sedgwick and Štefankovič use a similar approach to the one we use here in §5.6 of [26]; the form of the result givenin [3], though, is most convenient for our purposes.

Proposition 6.2 (Corollary 6.11 [3])**.** *Let M be a compact 2-manifold with triangulation T. Let γ be a connected normal curve in T, represented by its unsigned normal coordinates. There is an algorithm to compute the signed normal coordinates of some orientation of γ in time polynomial in $|T|$ and $\log(s(\gamma))$, where $s(\gamma)$ is the number of normal arcs in γ.*

Corollary 6.3. *Let $\gamma_1, \dots, \gamma_n$ be a collection of connected normal curves in a triangulation T of a compact 2-manifold, with an orientation of the surface that is given by an orientation of each triangle. We can compute the algebraic intersections of each pair of curves for some (fixed) choice of curve orientations in polynomial time in $|T|$ and $\sum_{i=1}^{n} \log(|\gamma_i|)$.*

Proof. Compute the signed normal coordinates of each curve and thus fix orientations on them. Fix a pair of curves, and for each edge of T, arbitrarily assign one end of the edge to each curve. Isotope the curve so that it intersects the edge only in that half, and draw each elementary segment as a straight line arc within each triangle. As described in [3, Corr. 6.12] and [26], we can then compute the number of positive and negative crossings of the pair within each triangle in constant time by multiplying at most six pairs of signed normal coordinates. $\square$

Thanks to work of Agol, Hass and Thurston, there is a polynomial time algorithm that, given a normal surface in a triangulation as a vector, computes its homeomorphism type. Lackenby, Haraway and Hoffman use it to quickly cut triangulations along normal surfaces.

Proposition 6.4 (Corollary 17 [1])**.** *Let M be a 3-manifold with a triangulation $\mathcal{T}$ and let F be a normal surface in M. There is a procedure for counting the number of components of F and determining the topology of each component that runs in time polynomial in $|\mathcal{T}| \log w(F)$.*

Proposition 6.5 (Proposition 13 [6])**.** *There is an algorithm that takes as its input both a compact connected orientable triangulation $\mathcal{T}$ and a connected normal surface S in $\mathcal{T}$ given as a vector, and provides as its output a triangulation of an exterior of S whose size is bounded by a polynomial in $|\mathcal{T}|$, $\log w(S)$, and the Euler characteristic of S, and runs in time polynomial in those same three parameters.*

Corollary 6.6. *There is an algorithm that takes as its input both a compact connected orientable triangulation $\mathcal{T}$ of a 3-manifold M and a (possibly disconnected) normal surface S in $\mathcal{T}$ given as a vector, such that no two components of S are normally isotopic, and provides as its output a triangulation of $M \backslash\backslash S$ whose size is bounded by a polynomial in $|\mathcal{T}|$, $\log w(S)$, and the minimal Euler characteristic of the components of S, and runs in time polynomial in those same three parameters. It also produces normal surface vectors for each component of the boundary of the copy of the double of S in $\partial(M \backslash\backslash S)$.*

Proof. Note that by Kneser-Haken finiteness there are at most $61|\mathcal{T}|$ components of S. Cut along S, applying Proposition 6.5 at each step. The construction of the triangulation in Proposition 6.5 follows Lackenby's argument in Theorem 9.2 of [17], which is in the setting of sutured manifolds, and keeping track of the boundary of the surface we cut along rather than of some new sutures does not significantly modify the argument. We perform $O(|\mathcal{T}|)$ steps, each of which takes time that is at most polynomial in $|\mathcal{T}|$, $\log w(S)$ and the minimal Euler characteristic of the components of S. $\square$

Proposition 6.7. *Deciding whether a 3-manfold M is an orientable circle bundle over a surface with non-empty boundary is in* ***NP****, and giving the homeomorphism type of the surface is in* ***FNP****. Furthermore, unless $M \cong K \tilde{\times} I$ or $T^2 \times I$, there exists a normal section and one normal fibre on each boundary component, such that these properties can be certified in time polynomial in $|\mathcal{T}|$.*

Haraway and Hoffman have previously shown that certifying $S \tilde{\times} S^1 \cong K \tilde{\times} I$ and $A \times S^1 \cong S \tilde{\times} I$ is in NP [6].

Proof of Proposition 6.7. Let $\mathcal{T}$ be a triangulation of M. The data given in the certificate is the following:

(1) A compatible orientation of each tetrahedron;
(2) If M is the solid torus, a fundamental normal disc and a curve in ∂M intersecting it once;
(3) If M is $T^2 \times I$, a normal annulus F of weight at most exponential in a polynomial of $|T|$, a triangulation of $M \backslash\backslash F$ constructed using Corollary 6.6, and a fundamental normal essential disc in this triangulation;
(4) If M is $K \tilde{\times} I$ (which we consider to have Seifert structure $S \tilde{\times} S^1$), where S is the Möbius band, a fundamental annulus F, and:
    (a) If the fundamental annulus is horizontal, the triangulation of $M \backslash\backslash F \cong S^1 \times D^2 \sqcup S^1 \times D^2$ from Corollary 6.6 and a fundamental normal essential disc in each component of the result;
    (b) If the fundamental annulus is vertical, the triangulation of $M \backslash\backslash F \cong S^1 \times D^2$ from Corollary 6.6 and a fundamental normal essential disc in it;
(5) Otherwise:
    (a) a collection of non-isotopic normal vertical annuli F, of total weight at most exponential in $|\mathcal{T}|^2$, whose complement is a solid torus;
    (b) the triangulation of $M \backslash\backslash F$ from Corollary 6.6;
    (c) a fundamental normal essential disc in this triangulation of $M \backslash\backslash F$;
    (d) a fundamental normal section.

Claim 1: The data of this certificate exists and has size bounded by a polynomial in $|\mathcal{T}|$.
Proof: When M is a solid torus, the data exists by Lemma 6.1.

When M is $T^2 \times I$, by Lemma 5.6 there is a normal annulus of the required size. Corollary 6.6 allows us to construct a triangulation of $M \backslash\backslash F$ of polynomial size in $|\mathcal{T}|$ as the weight of F is at most exponential in $|\mathcal{T}|$. By Lemma 6.1 then there is a fundamental normal essential disc in this new triangulation.

When M is $K \tilde{\times} I$, by Lemma 5.6 there is a fundamental normal essential annulus, which (as discussed in the proof of Proposition 5.7) is either horizontal and cuts M into two solid tori, or is vertical and cuts it into one solid torus. Either way we can use Corollary 6.6 to construct a triangulation of its complement that is of polynomial size in $|\mathcal{T}|$, and by Lemma 6.1 there is a fundamental normal essential disc in each component of this new triangulation.

In the general case, by Corollary 5.9 there is a collection of vertical essential annuli C in $\mathcal{T}$, of total weight at most $c_{\mathcal{T}}^{|\mathcal{T}|^2}$, whose complement is a single solid torus. By Corollary 6.6 we can construct a triangulation of their complement, recording the boundary of C, that is of size at most polynomial in $|\mathcal{T}|$. By Lemma 6.1 there is a fundamental normal essential disc in this new triangulation. By Proposition 3.7 there is a fundamental section in M. ∎

Claim 2: The homeomorphism type of M can be verified in polynomial time.
Proof: Check that M has boundary; that is, that some face of the triangulation is not identified with any other. Check that the given orientations of the tetrahedra are compatible, and thus certify that M is orientable.

If M is a solid torus, it is known by work of Ivanov that recognising it is in **NP** [10]. To certify the data from Lemma 6.1, check that the given surface is in fact a disc using Proposition 6.4 then check (using Corollary 6.3) that the algebraic intersection number of the curve and the boundary of the disc, given as normal curves, is ± 1. We thus know that the curve and disc are essential, so must be a fibre and meridian disc.

If M is $T^2 \times I$, certify that F is an annulus using Proposition 6.4. Apply Corollary 6.6 to produce a triangulation of $M \backslash\backslash F$ and normal curves in the triangulation from ∂F. We have already seen that we can certify that $M \backslash\backslash F$ is a solid torus. Compute the algebraic intersection numbers of the boundary curves from F with the boundary of the meridian disc, which (as the meridian curve is from a fundamental normal surface so is of at most exponential weight in $|\mathcal{T}|^2$) we can do in polynomial time by Corollary 6.3. Verify that they are each ± 1, so the core curve of F in $M \backslash\backslash F$ is a longitude. The mapping class group of the annulus up to non-∂-preserving isotopy has two elements: the class of the identity, and the class that exchanges the two boundary components, so M is either (in the first case) $T^2 \times I$ or (in the latter) $K \tilde{\times} I$. The first homology of $T^2 \times I$ is $\mathbb{Z}^2$ and that of $K \tilde{\times} I$ is $\mathbb{Z} \times \mathbb{Z}_2$, so it suffices to compute the homology of M, which, as the dimension of M is fixed, can be done in polynomial time by work of Iliopoulos [9] as explained in [13].

Suppose M is $K \tilde{\times} I \cong S \tilde{\times} S^1$ where S is the Möbius band. If the essential annulus F is horizontal it covers the Möbius band, so as it has two boundary components it is a degree two cover and separates M into two copies of $S \tilde{\times} I$ (where it is the horizontal boundary of this bundle), which is the solid torus. As in the $T^2 \times I$ case, we can efficiently certify that this fundamental surface separates M into two solid tori. We can use the normal curve vectors for the boundary of F and the boundaries of the meridian discs to verify with Corollary 6.3 that these curves have algebraic intersection number ± 2. Up to choice of coordinates there is only one curve on the boundary of the solid torus that intersects a meridian disc twice, so this is enough to certify that $\mathcal{T}$ is a triangulation of $K \tilde{\times} I$.

If F is vertical, it sits over a spanning arc in the Möbius band and cuts M into a single solid torus. As in the $T^2 \times I$ case, certify that $M \backslash\backslash F$ is a solid torus, and produce a normal vector for a meridian curve. Compute the algebraic intersection numbers of the boundary of the two copies of F with this meridian curve. We find that they are ± 1 and by the same homology computation as in the $T^2 \times I$ case we can certify that M is $K \tilde{\times} I$.

We are left with the general case. By Proposition 6.4 we can quickly check that each surface in F is an annulus and compute the number of components n in F. As we have already described, we can certify that $M \backslash\backslash F$ is a solid torus and that the given surface is indeed a meridian disc, and record the corresponding meridian curve. Compute the intersection number of each annulus boundary and this meridian curve (using Corollary 6.3) to certify that the annulus core curves are longitudes in the solid torus. Now, when we glue up the annuli we will get a circle bundle over a surface Φ. This surface has Euler characteristic $1 - n$. Note that as M is not a circle bundle over a disc, annulus or Möbius band, $1 - n$ is negative.

Let Σ be the normal surface that we claim is a normal section. Verify (using Proposition 6.4) that $\chi(\Sigma) = 1 - n$. Compute algebraic intersection numbers (using Corollary 6.3) to certify that the boundary curve of Σ intersects each boundary curve of each annulus in A exactly once, which implies that Σ intersects each annulus in one spanning arc and possibly some trivial curves. This shows that, after compressions and ∂-compressions (which increase Euler characteristic), Σ is horizontal; as $\chi(\Phi) < 0$ and $\chi(\Sigma) = \chi(\Phi)$, Σ must be a degree one horizontal surface: that is, a section. ■

For the collection of fibres, we have a normal section in the surface bundle and a complete collection of vertical essential annuli. Take a minimal collection of essential annuli that link all the boundary components of M', and take one annulus boundary component on each boundary component of M. □

7. Recognising Seifert fibered spaces with boundary is in NP

Proposition 7.1. *Deciding if a 3-manifold M with triangulation $\mathcal{T}$ is a Seifert fibered space with only multiplicity two singular fibres and non-empty boundary, other than $S^1 \times D^2$, $S^1 \times S^1 \times I$ and $K \tilde{\times} I$, is in **NP**. When M is such a Seifert fibered space, giving its Seifert data is in **FNP** and there is a degree two normal horizontal surface and one normal fibre on each boundary component such that these properties can be certified in polynomial time.*

Proof. The data given in the certificate is the following:

(1) the Seifert data of M;
(2) two collections of disjoint normal essential vertical annuli C in $\mathcal{T}$, of total weight at most $c^{|\mathcal{T}|^2}$, for a fixed constant c, where the complement of C is the union of a solid torus neighbourhood of each singular fibre and a circle bundle over a surface;
(3) a triangulation of $\mathcal{T}\backslash\backslash C$ from Corollary 6.6, with a record of the normal curves from the boundaries of the components of C;
(4) the data from the certificate in Proposition 6.7 for $\mathcal{T}\backslash\backslash C$;
(5) a fundamental normal essential disc in each solid torus component of $\mathcal{T}\backslash\backslash C$;
(6) a degree two horizontal fundamental normal surface F in $\mathcal{T}$;

Claim 1: The data of this certificate exists and is of size at most polynomial in $|\mathcal{T}|$.
Proof: Corollary 5.9 gives us the collection of normal vertical annuli. Use Corollary 6.6 to build the required triangulation, and then by Lemma 6.1 we can find a fundamental normal essential disc in each solid torus component of this triangulation. Proposition 6.7 gives us the certificate for $\mathcal{T}\backslash\backslash C$. By Proposition 3.7, the desired degree two horizontal fundamental normal surface exists. ■
Claim 2: The homeomorphism type of M can be verified from the certificate in polynomial time.
Proof: As in Proposition 6.7, certify that M has boundary and is orientable.

Build the triangulation of $\mathcal{T}\backslash\backslash C$ using Corollary 6.6, verifying that it agrees with the one in the certificate. We can then certify that all but one of the resulting pieces are solid tori by work of Ivanov [10], and certify the homeomorphism type of the remaining piece M' using Proposition 6.7. By computing algebraic intersection numbers using Corollary 6.3, we can check that for each of the solid torus meridian discs, each annulus (algebraically) intersects it twice or not at all. We thus know that M is a copy of M' with solid tori glued on by gluing a $(1, 2)$ slope curve in the boundary of the solid torus to a fibre of M'; this is enough to certify that these solid tori are neighbourhoods of singular fibres of multiplicity two. ■

We know that the boundary curves of the annuli are fibres by construction, so take one on each boundary component. It remains to certify that F is a degree two horizontal surface. Check that the algebraic intersection number of ∂F with each annulus is ± 2 using Corollary 6.3, and deduce that F compresses and ∂-compresses to a degree two horizontal surface by Lemma 3.6, so $\chi(F) \leq 2\chi(\Sigma) - n$. Now check that $\chi(F) = 2\chi(\Sigma) - n$, so we could not have done any compressions or boundary compressions since they increase Euler characteristic. □

We are now almost ready to prove these recognition results for general Seifert fibered spaces with boundary: that is, when we have singular fibres of multiplicities other than two. We need two more results: a theorem from previous work of the author and a result about computing singular fibre data in Seifert fibered spaces.

Theorem 7.2 (Theorem 1.2 [11])**.** *Let M be a Seifert fibered space with non-empty boundary and let $\mathcal{T}$ be a (material) triangulation of M. The collection of singular fibres of M that are not of multiplicity two have disjoint simplicial representatives in $\mathcal{T}^{(79)}$, the 79^{th} barycentric subdivision of $\mathcal{T}$. In $\mathcal{T}^{(82)}$, these simplicial singular fibres have disjoint simplicial solid torus neighbourhoods such that there is a simplicial meridian curve of length 48 for each such neighbourhood.*

Lemma 7.3. *Let M be an oriented Seifert fibered space with n of its singular fibres drilled out. Let F be a degree k horizontal surface in M, and let η_i, $1 \leq i \leq n$ be the collection of its curves of intersection with the boundary component arising drilling out the i^{th} singular fibre. Let γ_i, $1 \leq i \leq n$, be a regular fibre on each of these boundary components. Pick orientations of η_i and γ_i such that the algebraic intersection number $i(\eta_i, \gamma_i)$ is positive with respect to the orientation of ∂M induced by the orientation of M. Then if μ_i is a meridian from the drilled out solid torus, the Seifert data of this singular fibre q/p (with respect to this orientation of M and the basis of $H_1(\partial M, \mathbb{Z})$ induced by $[\frac{1}{k}\eta_i]$ and γ_i for each i) is $(i(\eta_i, \mu_i)/k)/i(\gamma_i, \mu_i)$.*

Proof. Note that if we flip the orientation of both η_i and γ_i, this ratio of intersection numbers does not change sign.

Note that η_i is k copies of a curve, and $(\frac{1}{k}\eta_i, \gamma_i)$ is a positive basis for the homology of this torus, with its induced orientation. By the definition of the construction of Seifert fibered spaces (for example, see [20, Defn. 10.3.1]), a (p_i, q_i) singular fibre is when the meridian curve is $\frac{p_i}{k}\eta_i + q_i\gamma_i$. □

Recall the results we wish to prove.

Theorem 1.1. *The problem* Seifert fibered space with boundary recognition *is in* ***NP****.*

Theorem 1.2. *The problem* naming Seifert fibered with boundary *is in* ***FNP****.*

Proof of Theorems 1.1 and 1.2. Let M be a Seifert fibered space with non-empty boundary. Our certificate will be of the following form. If M is a circle bundle over a surface or a Seifert fibered space with only multiplicity two singular fibres it will be the relevant data from Proposition 6.7 or Proposition 7.1, respectively. Otherwise, it will be:

(1) the Seifert data of M;
(2) a compatible orientation of each tetrahedron of $\mathcal{T}$;

(3) the triangulation $\mathcal{T}^{(82)}$, constructed by subdividing each tetrahedron of $\mathcal{T}$ in the order given;
(4) the non-multiplicity-two singular fibres in $\mathcal{T}^{(82)}$, a solid torus neighbourhood of each one, a meridian disc for it with boundary of length 48, a longitude curve in its boundary that intersects the meridian once, a triangulation $\mathcal{T}'$ of $\mathcal{T}^{(82)}$ with these neighbourhoods of singular fibres removed and length 48 meridian curves marked and a compatible choice of orientations of the tetrahedra of $\mathcal{T}'$;
(5) if there are singular fibres of multiplicity two, the certificate from Proposition 7.1 for $\mathcal{T}'$, or otherwise, the certificate from Proposition 6.7 for $\mathcal{T}'$.

It is straightforward to see that the certificate exists, as giving a triangulation of $\mathcal{T}^{(82)}$ is constructive, and Theorem 7.2 gives us the required singular fibre neighbourhoods.

To verify the certificate, first, as discussed in the proof of Proposition 6.7, check that M is orientable and has boundary.

Construct $\mathcal{T}^{(82)}$ by barycentrically subdividing the tetrahedra in order and verify that it agrees with the given triangulation. Certify that the removed regions are solid tori using Proposition 6.7 and use Proposition 6.4 to verify that the given meridian discs are indeed discs. Check that the longitude intersects the meridian curve once for each singular fibre neighbourhood using Corollary 6.3, thus certifying that the given meridian discs are essential.

Note that $\mathcal{T}' \not\cong K \tilde{\times} I$ as M' has more than one boundary component. Also, $\mathcal{T}' \not\cong T^2 \times I$ as the only Seifert fibered structure for $T^2 \times I$ is as $A \times S^1$, where A is the annulus, but then we can only have removed one singular fibre, so M was a solid torus to begin with. As we have the data of Proposition 6.7 or Proposition 7.1 to certify the Seifert data of $\mathcal{T}'$, since $\mathcal{T}' \not\cong K \tilde{\times} I$ or $T^2 \times I$, take the normal horizontal surface and complete collection of fibres in the boundary contained in this certificate. Note that the weight of this surface and annulus is at most $c_T^{|\mathcal{T}'|^2}$, where c_T is at most $10^{10^{36}}$. In each boundary component of M' that bounds a singular fibre, consider the boundary of the horizontal surface, η, and the boundary of a normal vertical annulus fibre, γ, each of whose length is at most $c_T^{|\mathcal{T}|^2}$. Push the simplicial meridian ν off the 1-skeleton to get a normal meridian μ. The length of μ is at most the length of ν plus the total valence of the vertices of this boundary torus: that is, at most three times the number of edges in this boundary torus, which is at most $3 \cdot 24^{82} \cdot 6|\mathcal{T}|$.

Check that the given orientation of the tetrahedra of $\mathcal{T}'$ is in fact consistent and compute the orientation of each boundary triangle induced by it. By Corollary 6.3, we can arbitrarily orient each of these three curves and then compute $i(\eta, \gamma)$ in polynomial time in the original input, with respect to the induced orientation of the boundary triangles. Set the orientation of η such that $i(\eta, \gamma)$ is positive and then compute $q' = i(\mu, \eta)$ and $p = i(\mu, \gamma)$. If $\mathcal{T}'$ had multiplicity two fibres, note that η is two copies of a horizontal curve and intersects γ twice, so set $q = \frac{q'}{i(\eta,\gamma)}$. By Lemma 7.3 this singular fibre has Seifert data q/p. With the certificate for M', this certifies the homeomorphism type of M, and we can check that this matches the given Seifert data. $\square$

Appendix A. Normal surfaces in split handle structures

A.1. Normalisation.

Lemma A.1. *Let S be an incompressible (connected) surface in the orientable I-bundle $\Sigma \overset{(\sim)}{\times} I$, other than a trivial disc or sphere, that is disjoint from the horizontal boundary and does not admit any ∂-compression discs with respect to the vertical boundary of $\Sigma \overset{(\sim)}{\times} I$. Then S is isotopic to $\Sigma \times \{*\}$ or its double cover.*

Proof. First, suppose that Σ is a disc. Then as $\Sigma \times I$ is a ball (so S is two-sided) and S is incompressible and hence π_1-injective, S is a sphere or a disc. If it is a sphere, it is a trivial one. If ∂S contains a trivial curve in the vertical boundary, then S must be a boundary-parallel disc. Thus S is a disc and its boundary is essential in the vertical boundary, so it is isotopic to $\Sigma \times \{*\}$.

For the general case, suppose that Σ has boundary. Decompose Σ into one 0-handle, which we view as a polygon, and some number of 1-handles, which we view as rectangles. Above each edge of the 0-handle and 1-handles in $\Sigma \overset{(\sim)}{\times} I$ is a quadrilateral with two boundary edges in the vertical boundary of $\Sigma \overset{(\sim)}{\times} I$ and two in the horizontal boundary. Isotope S so that it is transverse to these quadrilaterals and thus intersects each quadrilateral in a collection of arcs and closed curves. As S is incompressible, each of these closed curves bounds a disc in S, so by minimising the number of components of intersection between S and the quadrilaterals we can assume that the intersection is only of arcs. Note that S is disjoint from the horizontal boundary, so each of these arcs starts and ends on either the same or different vertical boundary arcs of a quadrilateral. If one starts and ends on the same vertical boundary arc, by taking an outermost such arc we can obtain a ∂-compression disc for S with respect to the vertical boundary, so by an isotopy we can remove this arc. Thus up to isotopy S intersects each quadrilateral in arcs that are transverse to the product structure.

Consider one of the handles of Σ, H, which is a disc. Consider the intersection of S with $H \overset{(\sim)}{\times} I \cong D^2 \times I$ in $\Sigma \overset{(\sim)}{\times} I$. We can assume (by minimising the number of components of intersection between S and the quadrilaterals) that S is incompressible in this copy of $D^2 \times I$, so as we have already discussed, each component of it is a (trivial) sphere (which we have ruled out) or disc. The boundary of $H \overset{(\sim)}{\times} I$ has three parts: its horizontal boundary, from which S is disjoint, quadrilaterals, and pieces of $\partial_v(\Sigma \overset{(\sim)}{\times} I)$. For similar reasoning as with the quadrilaterals, up to isotopy S also intersects the vertical boundary pieces in arcs that are transverse to the product structure. Thus the boundary of each disc of $S \cap H \overset{(\sim)}{\times} I$ is an essential curve in $\partial_v(H \overset{(\sim)}{\times} I)$. Thus S intersects $H \overset{(\sim)}{\times} I$ in a collection of horizontal discs for each H, and so intersects all of $\Sigma \overset{(\sim)}{\times} I$ in either $\Sigma \times \{*\}$ or (if Σ is nonorientable) possibly $\Sigma \tilde{\times} S^0$.

If Σ is closed, let D be a disc of Σ. Isotope S to minimise the number of components of $S \cap \partial(D \times I)$. The intersection of S with $D \times I$ is incompressible since otherwise, as S is incompressible, it would not be minimal. For the same reason $S \cap (D \times I)$ contains no spheres or trivial discs. By the first part of this proof, $S \cap (D \times I)$ is a collection of discs of the form $D \times \{*\}$. If $S \cap (\Sigma - D) \times I$ admits a ∂-compression disc with respect to the vertical boundary, we can use it to isotope S to reduce $|S \cap \partial(D \times I)|$. Thus by the previous part of the proof,

$S \cap (\Sigma - D) \times I$ is isotopic to $(\Sigma - D) \times \{*\}$ or its double cover, which gives us the result. □

We wish to modify an arbitrary surface F in M, whose boundary is disjoint from the forbidden region, by a series of admissible isotopies and normalisation moves such that the result is normal (but may not be isotopic to F). Compare the following procedure to the proof of Theorem 3.4.7 in [21]. The only substantial difference is in Move 2, as we need to consider parallelity pieces that are more complicated than 2-handles.

Procedure A.2 (Normalisation)**.** Let F be a properly-embedded surface in M that is disjoint from the forbidden region. The *weight* of F is $w(F) = (p(F), b(F), |F \cap \partial M|)$, which we will sort lexicographically, where $p(F)$, the *plate degree*, is $|F \cap (\partial\mathcal{H}^2 \cup \partial\mathcal{H}^{\mathcal{P}})|$ and $b(F)$, the *beam degree*, is $|F \cap \mathcal{H}^1|$. We will see in Proposition A.3 that almost all the *normalisation moves* which follow will reduce $w(F)$. All isotopies in these moves are required to be admissible.

The *normalisation procedure* is to perform Move 1 once, and then repeat Moves 2-7 in sequence as long as possible.

Move 1. Note that the boundary of the forbidden region $\mathcal{I}$ is disjoint from the boundaries of the 2-handles. (Admissibly) isotope F so that F is transverse to the handle structure and ∂F is disjoint from the horizontal boundaries of the 2-handles. Each 3-handle contains an open ball that is disjoint from F, so by expanding these balls, we can isotope F to be disjoint from the 3-handles. Discard any components of F that are entirely contained in a parallelity piece.

Move 2. Consider each component $H \cong \Sigma \overset{(\sim)}{\times} I$ of $\mathcal{H}^2 \cup \mathcal{H}^{\mathcal{P}}$. If any of the components of $F \cap \partial H$ are trivial curves in the vertical boundary, compress F along the discs in ∂H that these curves bound and isotope this part of F off ∂H. Similarly, if $F \cap H$ admits a compression disc, compress F along it. If $\partial_v H$ intersects ∂M (in which case H is a parallelity piece), and $F \cap H$ admits a ∂-compression disc D that is disjoint from the horizontal boundary – that is, where the arc of ∂D in ∂H, α, is in $\partial_v H$ – then compress along it. Now, as each component of $F \cap \partial H$ is an essential curve in $\partial_v H$, up to an isotopy supported in a collar of $\partial_v H$ within H, we can arrange that each of these curves is transverse to the induced I-bundle structure on $\partial_v H$.

If we performed a ∂-compression, repeat the first two steps of this move. Finally, discard any components of $F \cap (\mathcal{H}^2 \cup \mathcal{H}^{\mathcal{P}})$ that are spheres or $\partial_v H$-parallel discs.

Move 3. For each 1-handle $D^1 \times D^2$, take a disc $D = \{*\} \times D^2$, transverse to F, that minimises the number of components in $D \cap F$. We can blow a regular neighbourhood of D out to be the whole 1-handle.

Move 4. If any component of intersection of F with a 1-handle is a tube $S^1 \times I$, compress it and isotope the two resulting discs out of the 1-handle. Similarly, if any component of intersection with a 1-handle $D^1 \times D^2$ is a disc whose intersection with $D^1 \times \partial D^2$ is contained in a single region of $(\partial M - \mathcal{I}) \cap (D^1 \times D^2)$, we can ∂-compress this tunnel and isotope the pieces out of the 1-handle.

Move 5. Compress any compressible pieces of $F \cap \mathcal{H}^0$ and discard any trivial spheres in the 0-handles.

Move 6. If F intersects a lake in a trivial curve, compress it and throw away the resulting ∂-parallel disc. If F intersects a lake in an arc that starts and ends on the same component of the intersection of the lake with an island, by an isotopy (again as $\partial\mathcal{I}$ intersects the cell structure on the boundary in a normal curve) push this piece F off the lake and through the 1-handle.

MOVE 7. If a disc of $F \cap \mathcal{H}^0$ crosses a bridge twice or a bridge and an adjacent lake, isotope the portion of disk between them into the bridge. If it crosses a lake twice, ∂-compress the resulting tunnel along both of its intersections with the boundary of the lake.

Proposition A.3. *Let F be a properly-embedded surface in a split handle structure $\mathcal{H}$. Applying Procedure A.2 terminates and the result (if non-empty) is a normal surface.*

Proof. After Move 1 F satisfies condition 1 of Definition 4.4. If Move 2 has no effect, $F \cap H$ is incompressible and ∂-incompressible with respect to the vertical boundary, so by Lemma A.1 F satisfies condition 2 of Definition 4.4. If Moves 3 and 4 have no effect then it satisfies condition 4 of Definition 4.4 and condition 1 of Definition 4.5. After Move 5 it satisfies condition 5 of Definition 4.4. Then Move 6 ensures condition 2 of Definition 4.5 and Move 7 ensures condition 3. As a result, if no more moves can be performed, then F is normal.

Let the normalisation complexity of F be

$$(p(F), b(F), |F \cap \partial M|, \gamma(F), \eta(F), n(F)),$$

ordered lexicographically, where the new terms are $\gamma(F) = \sum_{i=1}^{m}(1 - \chi(F_i))$ where $\{F_i\}$ is the collection of connected components of $F \cap \mathcal{H}^0$ that are not spheres, $\eta(F) = \sum_{i=1}^{m}(1 - \chi(F_j))$ where $\{F_j\}$ is the collection of connected components of $F \cap (\mathcal{H}^2 \cup \mathcal{H}^{\mathcal{P}})$ that are not spheres, and $n(F)$ is the number of connected components of F. We will show that each of Moves 2-7 reduces the normalisation complexity of F.

For Move 2, suppose we perform a ∂-compression on $F \cap H$. As $\mathcal{H}$ is a split handle structure, if $\partial\mathcal{I}$ intersects a component of $\partial_v H \cap \partial M$ then it does so in two curves or arcs. Thus this component is either contained in $\mathcal{I}$, or (as $\partial_h H$ is contained in $\mathcal{I}$) it intersects $\mathcal{I}$ in a neighbourhood of its boundary with the horizontal boundary of H. If $F \cap H$ admits a ∂-compression disc D that is disjoint from the horizontal boundary, then let $\bar{\alpha}$ be a properly-embedded arc in $\partial_v H$, such that α is a subarc of it, which is isotopic in $\partial_v H$ to one of the I fibres of $\partial_v H$ and has minimal intersection with each of the components of $F \cap \partial_v H$ – that is, it intersects each of the essential curves in this collection once. This is possible since α runs between two different components of $F \cap \partial_v H$; otherwise we would be able to upgrade our ∂-compression disc to a compression disc. Note that $|\bar{\alpha} \cap F|$ is the degree of the projection map from $F \cap H$ to Σ. (When Σ is orientable, this is the number of sheets of F in H.)

The effect of the ∂-compression on $F \cap \partial H$ is to remove from it $\partial\alpha \times I$ and add to it $\alpha \times \partial I$, for some small thickening $\alpha \times I$ of α. This reduces $|\bar{\alpha} \cap F|$ by two, and thus we see that the number of essential curves in $F \cap \partial_v H$ has reduced by two, so either $p(F)$ has reduced or we reduce it in the next step of Move 2. After this move, $F \cap H$ does not admit any ∂-compression discs with respect to the vertical boundary of H.

Compressing F along trivial curves in the vertical boundary of H reduces $p(F)$. Compressing $F \cap H$ reduces $\eta(F)$ and fixes the earlier terms in the normalisation complexity. We can thus see that we either reduce $p(F)$ or reduce $\eta(F)$ and fix the earlier terms in the normalisation complexity.

If Move 3 is non-trivial then it does not increase $p(F)$ and reduces $b(F)$, and the same applies for Move 4. Move 5 does not change $p(F)$ or $b(F)$. If there are compressible pieces then it reduces $\gamma(F)$, and otherwise if all the pieces are 2-spheres or discs then it reduces $n(F)$ and fixes everything else. Move 6 either reduces $|F \cap \partial M|$ and does not change $p(F)$ and $b(F)$, or fixes $p(F)$ and reduces $b(F)$. Move 7 either decreases $p(F)$ or fixes it and reduces $b(F)$. As a result, the procedure terminates with a normal surface. □

Definition A.4. A properly-embedded disc in a 3-manifold M is *essential* if its boundary does not bound a disc in ∂M.

Note that this definition does not change if there is forbidden region.

Lemma A.5. *Suppose that M is irreducible with a split handle structure $\mathcal{H}$ such that there is some essential disc in M that avoids the forbidden region. Then there is a normal essential disc, disjoint from the forbidden region.*

Proof. Apply the normalisation procedure to this essential disc D. As M is irreducible and D is (trivially) incompressible, whenever we compress D in the normalisation procedure we will produce a surface admissibly isotopic to D and a trivial sphere, which we can discard. Whenever we ∂-compress D we will produce two discs, both of which are disjoint from the forbidden region. At least one of them must be essential as ∂D does not bound a disc in ∂M. Discard the other. □

Proposition A.6. *Let F be an incompressible ∂-incompressible properly-embedded surface in an irreducible ∂-irreducible manifold M with split handle structure $\mathcal{H}$, disjoint from the forbidden region, such that no component is a trivial sphere or disc or is entirely contained in a parallelity piece. Then F is admissibly isotopic to a normal surface.*

Proof. Apply the normalisation procedure to F. As F is incompressible and ∂-compressible and M is irreducible and ∂-irreducible, each time we (∂-)compress F in the normalisation procedure we will produce two components: a surface that is admissibly isotopic to F and a trivial sphere or disc. Thus we produce a normal surface admissibly isotopic to F, as well as some collection of trivial spheres and discs, which we can discard. □

A.2. **Subtetrahedral split handle structure combinatorics.** We give bounds on the number of elementary discs types.

Lemma A.7. *Let H be a subtetrahedral 0-handle. Let G be its boundary graph in $\partial H \cong S^2$. Then G contains between one and four islands; each island has at most three components of intersection with bridges; and if b is the number of bridges of G, then the number of sutures of G at most $12 - 2b$ and an island intersecting v bridges has at most $6 - 2v$ intersections with sutures.*

Proof. Consider a semitetrahedral handle structure $\mathcal{H}$ and normal surface F such that H is homeomorphic to one of the pieces of $\mathcal{H}\backslash\backslash F$ from some 0-handle H' in $\mathcal{H}$. We can think of forming the pieces of H' in the induced split handle structure as occurring in two steps. First, we cut H' along a collection of elementary discs. This has the effect on the boundary of H', which we think of as a graph embedded in S^2, of cutting it along a collection of (separating) curves and filling these holes in with the forbidden region. Second, we possibly replace some of the 1-handles with parallelity pieces, in which case (as, if a 1-handle can be given a parallelity structure, so can any 2-handles it borders) the effect on the boundary graph is to merge a valence one or two island with the bridge(s) adjacent to it.

Consider cutting along one elementary disc D of F in H' at a time. As $\mathcal{H}$ is semitetrahedral, the islands of H' are at most trivalent, so ∂D runs through each island at most once. As ∂D runs through each island or bridge at most once and is separating, the subgraph of islands and bridges after cutting along it is a subgraph of the complete graph on four vertices, so after cutting into pieces, there are between zero and four islands, each of which has valence at most three. We can rule out the case when the boundary graph is empty as, since the boundary graph of H' is connected, the boundary of any elementary disc runs through at least one island, so there will be an island in each of the pieces of H' that it separates. Turning 1-handles into parallelity handles does not increase the number of islands or their valence. If all of the 1-handles become parallelity pieces (so there were zero islands) then in fact the entire 0-handle will become a parallelity piece, so that is impossible.

Let G be the boundary graph of H' (containing b bridges) and $\mathcal{I}$ its forbidden region, where we assume that G is connected and contains at most $12 - 2b$ sutures. The elementary disc boundary ∂D separates $\partial H'$ into two components. Pick one of them, C, to consider. When we cut along D, the other component from $\partial H'$ has one less bridge than G for every bridge fully contained in C, one less suture for each suture in C, and one more suture for each arc of D that runs through a lake. Thus for the count of sutures it suffices to prove that twice the number of bridges fully contained in C plus the number of sutures of H' in C is at least the number of arcs in ∂D that run through a lake. Consider one of these arcs α. It separates the lake into two components, one of which is in C, and each of which is bordered by at least one bridge or suture. If the component in C is bordered by a bridge, assign the arc α to that bridge. Note that as the bridge is adjacent to at most two lakes, and ∂D crosses each lake only once, by doing this we will associate at most two arcs to the bridge. Also note that as the elementary disc does not run through an adjacent bridge and lake, this bridge is contained in C. Otherwise, this component in C is bordered by at least one suture, as α runs between two different components of intersection of an island and a lake, so assign α to this suture, which again we see is contained in C. The forbidden region is on the other side of this suture so we will not assign any other arcs to it. We can thus see that the count is as we claimed. Turning 1-handles into parallelity regions only reduces the number of bridges further.

For the final statement, fix an island in $\mathcal{H}$ of valence v that intersects at most $6 - 2v$ sutures. Consider some elementary disc boundary running through this island. By the same procedure as for the total suture count, assign any new sutures that are created to a bridge or a suture that intersects the island to get the result: we start with three adjacent bridges, and add at most two sutures each time we remove one. □

Notation A.8. Write $|\mathcal{H}|$ for the number of 0-handles in $\mathcal{H}$.

Lemma A.9. *The total number of bridges and lakes in a subtetrahedral trace 0-handle H is at most 13.*

Proof. From Lemma A.7, each 0-handle contains at most 6 bridges and $12 - 2b$ sutures, where b is the number of bridges. The number of lakes from the subgraph on islands and bridges alone is at most $b+1$ by an Euler characteristic argument. Each pair of sutures can increase the number of lakes by one, if they bound a forbidden region that borders a lake on both sides. (If a suture bounds a forbidden region that borders a bridge on one side then it does not increase the complexity.) Thus having sutures does not in fact increase the count, so that maximum count is $6+6+1=13$. □

Lemma A.10. *If F is a (possibly disconnected) non-duplicate normal surface in a subtetrahedral split handle structure $\mathcal{H}$ then $|\mathcal{H}\backslash\backslash F|$ is at most $2^{13}|\mathcal{H}|$.*

This is a substantial overestimate, but will do for our purposes.

Proof. We know that $\mathcal{H}\backslash\backslash F$ is a split handle structure from Lemma 4.9. Define the complexity of a subtetrahedral 0-handle H, $c(H)$, to be the sum of the number of lakes of H and the number of bridges. By Lemma A.9 $c(H)$ is at most 13. Consider an elementary disc D that separates H into two pieces. After we cut along D and take the induced handle structure, we obtain at most two 0-handles, H_1 and H_2. We show that $c(H_i)$ is at most $c(H)$ for each i, with equality only if the other piece is a parallelity piece.

As ∂D does not run through a lake or bridge twice, cutting along an edge of D cannot increase either of the terms in the complexity. The boundary of D separates the boundary of H and is not a trivial curve, so as it does not run through a bridge or lake twice or through a bridge and an adjacent lake, it must separate off at least one forbidden region, bridge, or lake on both sides. If it separates off exactly one forbidden region and no bridges or lakes on one side, then it is parallel to the boundary of the forbidden region so one of the H_i is a parallelity handle. If it separates off more than one forbidden region then two of those forbidden regions had a lake or bridge between them, so it has reduced the number of lakes. If it separates off a bridge or lake on one side, then cutting along it reduces the number of bridges or lakes.

Thus each time we cut a 0-handle H into pieces, if we produce two pieces then they both have lower complexity than H, so we may obtain at most 2^{13} 0-handles in $\mathcal{H}\backslash\backslash F$. □

Lemma A.11. *There are a finite number of subtetrahedral 0-handles up to homeomorphism, and each of these can contain a finite number of elementary disc types up to normal isotopy.*

Proof. That there are a finite number of types of subtetrahedral 0-handles follows from the combinatorial constraints of Lemma A.7. An elementary disc D in such a 0-handle H is determined by its boundary ∂D, which is itself completely described (as the complement of the boundary graph is a collection of discs) by the ordered list of islands it runs through and, for each, which intersection with a bridge or lake it enters and leaves by. As it may run through each island, bridge or lake at most once, there are a finite number of possibilities. □

Notation A.12. Write h_H for the number of possible subtetrahedral 0-handle types, and write d_H for the maximum possible number of elementary disc types up to normal isotopy in a subtetrahedral 0-handle.

Remark A.13. By Lemma A.9, each 0-handle has a total of at most 13 lakes and bridges, so, by labelling each disc type by the number of edges in it and then the order of them, d_H is at most $13 \cdot 13!$. This bound is far from sharp; enumerating by hand shows that the true number for a tetrahedral 0-handle (whose boundary graph is the complete graph on four vertices) is 59 elementary disc types, so the author would be quite surprised if d_H was above 200.

Note that there is a natural inclusion map from surfaces in $M \backslash\backslash F$, disjoint from the forbidden region, to surfaces in M, which takes normal surfaces to normal surfaces.

A.3. **Summands of incompressible normal surfaces.** In this section all split handle structures are subtetrahedral. In Matveev's excellent book [21], he carefully lays out the foundations of normal surface theory in the triangulation setting. Although we are in a more general setting, it is similar enough for the ideas in the proofs he gives to apply. See the proof of Theorem 4.1.36 in [21] for his treatment of these results in the triangulation setting, drawing on work of Haken, Jaco, Oertel and Tollefson, among others. The analogues of the structure he relies on are Lemmas 4.14 and 4.15, which show respectively that performing an irregular switch reduces the weight of a normal surface sum, and that no normal surface has zero weight. Recall the definition of the sum of normal surfaces in split handle structures in Procedure 4.12 and the definition of the weight of a normal surface in Definition 4.13.

Definition A.14. Suppose that $F = G_1 + G_2$. Let γ be the collection of double curves of $G_1 \cup G_2$. A *patch* of $G_1 \cup G_2$ is a component of $G_1 \cup G_2 - \gamma$.

Definition A.15. A surface F is *minimal* if it is of minimal weight in its admissible isotopy class.

Proposition A.16. *Let M be an irreducible ∂-irreducible 3-manifold and let $\mathcal{H}$ be a subtetrahedral split handle structure for M. Let F be a normal surface of minimal weight in its (admissible) isotopy class which is incompressible and ∂-incompressible, with $F = G_1 + G_2$. Then G_1 and G_2 are also incompressible and ∂-incompressible, and neither G_1 nor G_2 is a copy of S^2, $\mathbb{R}P^2$ or a disc.*

To set up, isotope G_1 and G_2 to minimise the number of components in $G_1 \cap G_2$ subject to the requirement that $F = G_1 + G_2$, so that $G_1 \cup G_2$ is in *reduced form.* Note that as M is irreducible and ∂-irreducible, no component of F is a disc or sphere, as then we could isotope it to the interior of a 0-handle and so, in fact, there was no minimal normal surface representative of this surface. A *disc patch* is a patch, homeomorphic to a disc, whose boundary is either a closed curve in the interior of M, or an arc in the interior and an arc on ∂M (which we note must be disjoint from the forbidden region).

To prove Proposition A.16, we first give a lemma that is analogous to Lemma 4.1.8 of [21].

Lemma A.17. *The patches of $G_1 \cup G_2$ are incompressible and ∂-incompressible, and none are disc patches.*

Proof. First, we show there are no disc patches. Suppose that a disc patch E did exist, say in G_1. Write s for its double curve in $G_1 \cup G_2$, and s_1 and s_2 for the corresponding trace curves in F, where s_1 bounds E and s_2 is the other curve. If the boundary of this patch is one-sided in G_1, then the connected component of G_1

containing E is a normal projective plane P. As M is irreducible, it is $\mathbb{R}P^3$ and P is incompressible. By Lemma 4.15 all normal surfaces in M have non-zero beam degree. In particular, neither $b(G_1)$ nor $b(G_2)$ is zero, so $b(P)$ is strictly less than $b(F)$ and $p(P)$ is at most $p(F)$. Now, $\mathbb{R}P^3$ contains (up to isotopy) only one closed incompressible surface that does not contain a sphere: this projective plane. Thus F was not minimal.

Suppose that s_2 does *not* bound a disc in F. But then taking a parallel copy of E, bounded by s_2, we get a compression or ∂-compression disc for F.

Suppose that s_2 bounds a disc E' in F that is opposite to E at the intersection curve. If E' does not contain E, then $E \cup E'$ is a sphere that (as M is irreducible) bounds a 3-ball, so E and E' are isotopic. We can take the irregular switch at s instead to obtain an isotopic surface to F which by Lemma 4.14 is of lower weight than F, so F was not minimal.

If E' does contain E, again taking the irregular switch along s, we get, if E was an interior disc, two surfaces $F_1 \cup T$. The surface F_1 is homeomorphic as a surface to F and $\partial F_1 = \partial F$, while T is formed by taking $E' \backslash E$ and identifying the boundary circles, and so is a torus or Klein bottle. As the switch was irregular, at least one of F_1 and T contains a return and hence after normalising has its weight decreased. Let F' be the result of normalising F_1. If the return is in F_1 then the weight of F' is less than that of F; if the return is in T then the weight of T is non-zero and so we can draw the same conclusion. As, since M is irreducible, E' is isotopic to a parallel copy of E, F and F' are isotopic and so F was not minimal.

If E intersected the boundary, we can follow a similar line of reasoning. Now after an irregular switch we get $F_1 \cup A$ where A is an annulus or Möbius band. The same reasoning however still holds.

The remaining case in which there are disc patches is when every disc patch has an adjacent companion disc. This is the case considered in Lemma 4.1.4 of [21], where Matveev shows the following result in the triangulation setting:

Suppose that every disc patch of $G_1 \cup G_2$ has an adjacent companion disc E'. Then either F can be presented as $F = F_1 + T$ where T has Euler characteristic zero and F_1 is admissibly isotopic to F and of lower weight, or there is a disc patch E whose adjacent companion disc is also a patch of F.

We give the idea of Matveev's proof, which also goes through in our setting. It is to associate to $G_1 \cup G_2$ an oriented graph, whose vertices are disc patches and there is an edge from E_1 to E_2 if E_2 is contained in the adjacent companion disc E'_1 of E_1. As every vertex has at least one outgoing edge, this graph contains a cycle. Resolve $G_1 \cup G_2$ with the irregular switch at the edges of the patches in this cycle, and the regular switch otherwise. We get a decomposition $F = F_1 + T$ where T is the union of the pieces $E'_i - E_{i+1}$. As by Lemma 4.15 T has non-zero weight, F_1 is of lower weight than F. Note that F is homeomorphic to F_1 as if we perform regular switches everywhere, then at ∂E_i we replace E_i with $A_i \cup E_{i+1}$. Now, $A_i \cup E_{i+1} \cup E_i$ is a disc or a sphere, so as M is irreducible and ∂-irreducible they separate off balls. We can use this to construct the desired isotopy.

Now, as F is minimal, there must be some disc patch E whose adjacent companion disc E' is also a patch. Let G'_1 and G'_2 be the surfaces resulting from swapping E and E' within G_1 and G_2. Now, $E \cup E'$ is either a sphere or a properly-embedded disc, so as M is irreducible and ∂-irreducible, we can isotope G_1 to G'_1 and G_2 to

G_2', reducing the number of intersections in $G_1 \cup G_2$. Thus $G_1 + G_2$ was not in reduced form, as we assumed.

We can therefore conclude that there are no disc patches. Suppose D is a compression disc for some patch P, which we perturb to be transverse to F. Let γ be an innermost curve of $F \cap D$, which, as F is incompressible, bounds a disc D' in F. If $P \subseteq D'$ then P is planar and all but one of its boundary components bound discs in F. But then either P is a disc patch or the disc bounded by one of these boundary components contains a disc patch. Thus D' is disjoint from P so we can therefore isotope D to remove all intersections with F other than its boundary. Now, ∂D bounds a disc in F. This disc must be contained in P as otherwise some component of ∂P bounds a disc, and so there is a disc patch. Thus P is incompressible.

Suppose that D is a ∂-compression disc for P. By the same reasoning, we can isotope D to remove any curves of intersection with F. If α is an outermost arc of $F \cap D$ other than the segment of ∂D in P, as F is ∂-incompressible, it bounds a disc in F together with an arc in the boundary. By the same reasoning, this disc is disjoint from P so we can isotope D to further remove these arcs. Thus $D \cap P$ bounds a disc in F, which, as there are no disc patches, is contained in P. $\square$

Proof of Proposition A.16. As any decomposition of a disc, sphere or projective plane has a disc patch, each component of G_1 and G_2 has nonpositive Euler characteristic. If G_1 is compressible or ∂-compressible, we will construct a nontrivial compression or ∂-compression disc D for G_1 that is disjoint from G_2. Let the *weight* of a compressing disc D for G_1 be $w(D) = |D \cap G_2| + |\partial D \cap (G_2 \backslash\backslash \partial M|$. The idea is to give moves to reduce this weight.

Claim 1: Suppose that G_1 has a nontrivial compression or ∂-compression disc D, which we can assume is transverse to G_2. If $D \cap G_2$ contains a closed curve or an arc with both endpoints in ∂M, then there is a compression or ∂-compression disc D' for G_1, of lower weight, that does not contain such intersections with G_2.

The triangulation setting analogue of this claim is shown in the first part of Lemma 4.1.35 of [21]. We sketch the idea of the proof, which does not in fact depend on its setting. Take an innermost such curve or outermost such arc, bounding a disc Δ in D such that $\Delta \cap G_2 = \partial\Delta$ or $\Delta \cap G_2$ is a single arc. As $\partial\Delta \cap G_2$ is connected, it is contained in a single patch of G_2. Now, as the patches of $G_1 \cup G_2$ are incompressible and ∂-incompressible and M is irreducible and ∂-irreducible, there is a disc in this patch of G_2 isotopic to Δ. We can replace Δ in D with this disc to remove this intersection with G_2 and reduce the weight of D.

Let D be a nontrivial compression or ∂-compression disc for G_1 that intersects G_2 minimally. By the claim, D intersects G_2 only in arcs that have at least one endpoint away from ∂M. Let Δ be a region of D cut out by $D \cap G_2$. Then Δ is a (∂)-compressing disc for the polyhedron $G_1 \cup G_2$, as it is transverse to the singular subcomplex of $G_1 \cup G_2$ and $D \cap \partial M$ is connected. Following [21, §4.1.6], label the vertices of Δ as "good" or "bad" angles according to whether, in F, the patches forming the two edges are pasted together or not respectively. Note that if all angles of Δ are good, then Δ is a compression disc for F.

The following claim is Lemma 4.1.33 of [21], and his proof goes through in our setting.

Claim 2: Suppose that Δ has exactly one bad angle. Then F is not minimal.

The idea of the proof is to consider the component l of $G_1 \cap G_2$ that passes through this bad angle. Let A be the annulus or quadrilateral joining the two copies of l in F. One can build a (∂-)compressing disc for F from, if l is a closed curve, two copies of Δ and one of the components of $A - \Delta$, or if l is an arc, Δ and a component of $A - \Delta$. Now as F is incompressible and ∂-incompressible and M is irreducible and ∂-irreducible, this disc bounds a ball with a disc of F and possibly a disc in ∂M. We can use this to isotope F through a region it bounds with A to a surface of lower weight.

By this claim, each region has either no bad angles or at least two bad angles. Now, each arc in $D \cap G_2$ contributes one bad angle for each endpoint it has that is not on ∂M. Let m be the number of arcs that have no endpoints on ∂M, and n be the number with one endpoint on ∂M. There are $m + n + 1$ regions in D, and $2m + n$ bad angles. Thus, by the pigeonhole principle, there is a region Δ with at most one bad angle, which hence has no bad angles.

This region is a compression or ∂-compression disc for F itself, which is incompressible and ∂-incompressible, so $\Delta \cap F$ bounds a disc Δ' in F. We sketch the ideas of Lemma 4.1.34 of [21], which deals with this situation. First, Δ' is decomposed by the trace curves into smaller regions. As there are no disc patches, none of these trace curves are closed curves or are arcs with both endpoints on ∂M. Thus, taking an outermost trace curve, there is a region Δ'_0 of Δ' whose boundary is one trace curve segment, one segment of $\Delta \cap F$, and possibly one segment from ∂M.

Now, as shown in Lemma 4.1.35 of [21], if this region is in G_1, so $\partial \Delta'_0 = \Delta'_0 \cap (G_2 \cup \Delta \cup \partial M)$, then as M is irreducible we can isotope D through Δ'_0 to remove an arc of intersection with G_2. If this region is in G_2, then compress Δ along Δ'_0, removing an intersection. At least one of the resulting discs is essential and gives a non-trivial compression or ∂-compression disc. Either of these reduces the weight of Δ. Continuing this procedure, we eventually get a non-trivial compression or ∂-compression disc for G_1 that is disjoint from G_2. But then its boundary is contained in a patch of G_1, which is incompressible and ∂-incompressible, so this is a contradiction. $\square$

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