Triangulation Complexity of Hyperbolic Mapping Tori

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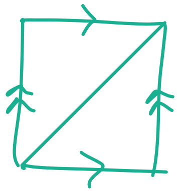
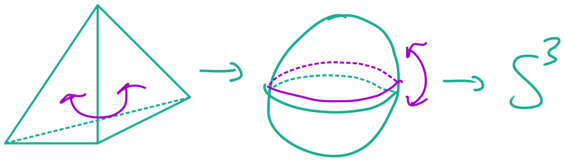
The triangulation complexity of a 3-manifold M, $\Delta(M)$, is the
compact, orientable
minimal number of tetrahedra in any triangulation of M.
pos. ideal, not necessarily simplicial

Ex.

(1) $\Delta(T^2) = 2$



(2) $\Delta(S^3) = 1$



(3) $\Delta(S^2 \times S^1) = 2$

Regina

(4) $\Delta(S^1 \times S^1 \times S^1) = 6$

Motivation

Enumerate all 3-manifolds

Computations

normal surface theory

unknot recognition

Snappea, Regina...

Rmk. For each n, only finitely many 3-manifolds with triangulation complexity at most n.

Goal: understand how $\Delta(M)$ relates to the geometry + topology of M.

Knot complements

crossing number

Want to relate $c(K)$ to $\Delta(S^3 - K)$

knot complement

$K \subset S^3$ is determined by its complement
(Gordon–Luecke)

Knot complements

Want to relate $c(K)$ to $\Delta(S^3 - K)$

① $\Delta(S^3 - K) \leq 4\,c(K)$ Snappea

D. Thurston

② $c(K) \leq \dfrac{\Delta(S^3-K)+1}{4} \cdot 545^{\Delta(S^3-K)}$

Haraway – Hoffmann – Schleimer – Sedgewick

Hyperbolic volume

Suppose M is closed and hyperbolic

i.e. $M \cong \mathbb{H}^3 / \Gamma$ ← discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$

By Mostow rigidity, $\mathrm{vol}(M)$ is well-defined.

Thm. (Gromov, Thurston)

= simplicial volume

$$f(\mathrm{vol}(N)) \neq \Delta(M) \geq \frac{\mathrm{vol}(M)}{\text{volume of a regular ideal tetrahedron in } \mathbb{H}^3}$$

Hyperbolic volume

We might wonder if $\Delta(M) \sim \mathrm{vol}(M)$.

Hyperbolic Dehn surgery

Given a finite volume hyperbolic manifold N with $\partial N = T^2$, get infinite family $\{N_i\}$ with almost all hyperbolic, such that $\mathrm{vol}(N_i) \leq \mathrm{vol}(N)$.

$$\Delta(M) \nleq f(\mathrm{vol}(M))$$

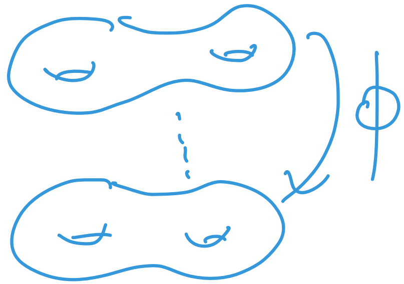
Mapping Tori

Thurston norm on $H_2(M_\phi)$

Let Σ be a closed surface

$\phi : \Sigma \to \Sigma$

The *mapping torus* M_ϕ is



$$\Sigma \times I / \phi$$

Futer-Schleimer:

How is the geometry of M_ϕ related to the dynamics of ϕ?

Mapping Tori

Thurston

Suppose M_ϕ is hyperbolic ($\iff$ ϕ pseudo-Anosov)

STL

Stable translation length of ϕ on X is

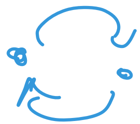
$$\inf\left\{ \frac{d(x,\phi^N(x))}{N} \;\middle|\; N \in \mathbb{Z}_+ \right\} \text{ for } x \in X$$

independent of $x \in X$

ℓ $\tilde{\ell}$

$\ell \leftarrow \phi_1$

$0 \leftarrow \phi_2$



Brock '03: $\mathrm{vol}(M_\phi) \sim$ STL of ϕ on pants graph $\sim \phi$ on $\mathcal{T}(\Sigma)$ with WP-metric

Mapping Tori

Suppose M_ϕ is hyperbolic ($\Longleftrightarrow$ ϕ pseudo-Anosov)

Stable translation distance of ϕ on X is $\inf\left\{ \frac{d(x, \phi^N(x))}{N} \,\middle|\, N \in \mathbb{Z}_+ \right\}$

Futer-Schleimer '14:

when Σ has ∂, vol(cusps in M_ϕ)

$\sim$ STL of ϕ in arc complex

Mapping Tori

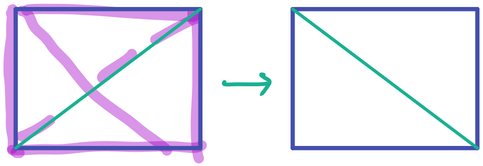
Suppose M_ϕ is hyperbolic ($\iff \phi$ pseudo-Anosov)

Lackenby-Purcell '19:

$\Delta(M_\phi) \sim$ STL graph of ϕ in triangulation

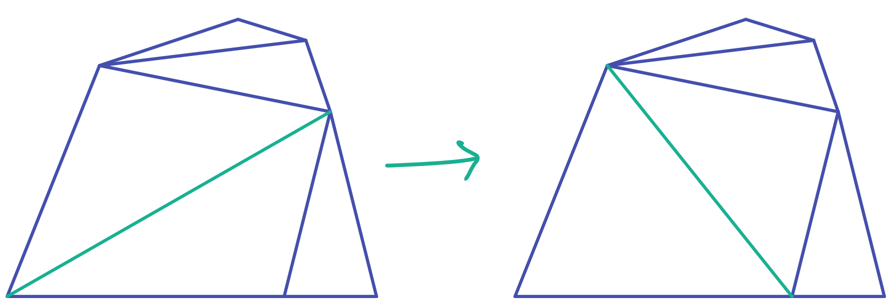
vertices: one vertex triangulations

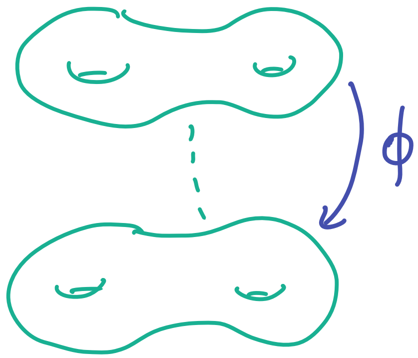
edges:



Idea of upper bound

Suppose $d(T, \phi(T))$ in triangulation graph is small





How does the STL of ϕ vary under composition?

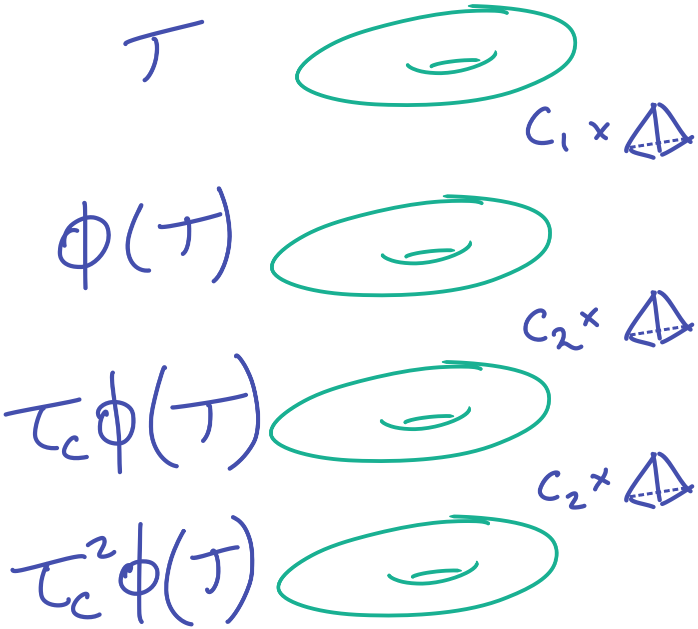
Prop. (J. '21) When $\phi : T^2 \to T^2$ is Anosov, for τ_c a Dehn twist,

① $\tau_c^n \phi$ is Anosov for all but finitely many n, and

② the STL of $\tau_c^n \phi$ in the triangulation graph grows linearly in n

How does the STL of ϕ vary under composition?

<u>Cor.</u> $\Delta\left(M_{\tau_c^n \phi}\right) \sim c_1 + n c_2$



ϕ pseudo-Anosov

traintrack

Agol

Bestvina-Handel

maximal splits

$$\tau_1, \tau_2, \dots, \tau_n, \dots, \tau_{n+k}$$

$$\tau_{n+k} = \phi(\tau_n)$$

Tillmann Rubinstein

Jaco